The Quadratic Formula

- The solutions to the equation \( ax^2 + bx + c = 0 \) \((a \neq 0)\) are:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

Proof of the quadratic formula:

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>(ax^2 + bx + c = 0)</td>
<td>Initial equation</td>
</tr>
<tr>
<td>(ax^2 + bx = -c)</td>
<td>Isolating the variable</td>
</tr>
<tr>
<td>(x^2 + \frac{b}{a}x = \frac{-c}{a})</td>
<td>Dividing by (a) so that (x^2) has a coefficient of 1</td>
</tr>
<tr>
<td>(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{-c}{a} + \frac{b^2}{4a^2})</td>
<td>We’re completing the square. The “b term” is (\frac{b}{a}). One–half of (\frac{b}{a}) is (\frac{b}{2a}). The square of (\frac{b}{2a}) is (\frac{b^2}{4a^2}). We add that to both sides.</td>
</tr>
<tr>
<td>(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a})</td>
<td>Switching the two terms on the right hand side of the equation (commutative property).</td>
</tr>
<tr>
<td>(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{\frac{b^2}{4a^2} - \frac{c}{a}}{a(4a)})</td>
<td>The common denominator of the right hand side is (4a^2). This means we have to multiply the top and bottom of the second term on the right hand side by (4a).</td>
</tr>
<tr>
<td>(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{\frac{b^2}{4a^2} - \frac{-4ac}{4a^2}}{4a^2})</td>
<td>Simplifying the second term on the right hand side.</td>
</tr>
<tr>
<td>(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2})</td>
<td>Since the two terms on the right hand side now have the same denominator, we can write them under that denominator.</td>
</tr>
<tr>
<td>(x + \frac{b}{2a})</td>
<td>The left hand side factors into ((x + \frac{b}{2a})^2) because it’s a perfect square and we know the factorization based on how we completed the square root.</td>
</tr>
<tr>
<td>(x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}})</td>
<td>By the square root property.</td>
</tr>
<tr>
<td>(x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a})</td>
<td>The square root of (4a^2) is (2a).</td>
</tr>
<tr>
<td>(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a})</td>
<td>Subtracting (\frac{b}{2a}) and combining since the denominators are the same</td>
</tr>
</tbody>
</table>
Solving Equations Using the Quadratic Formula:
1. Write the equation in standard form \( ax^2 + bx + c = 0 \).
2. Identify \( a \), \( b \), and \( c \).
3. Substitute those values into the quadratic formula.
4. Simplify.
5. Verify your solutions.

Example: Solve \( x^2 + 2x - 3 = 0 \) by all three methods we know.

I. Factoring.
\[(x + 3)(x - 1) = 0\]
\[-3, 1\]

II. Completing the square
\[
\begin{align*}
\frac{1}{2} (2) &= 1 \\
1^2 &= 1
\end{align*}
\]
\[
\begin{align*}
\frac{1}{2} \cdot 2 &= 1 \\
1^2 &= 1
\end{align*}
\]
\[
\begin{align*}
x^2 + 2x &= 3 \\
x^2 + 2x + 1 &= 3 + 1 \\
(x + 1)^2 &= 4 \\
x + 1 &= \pm \sqrt{4} \\
x + 1 &= \pm 2 \\
x &= -1 \pm 2 \\
\{-3, 1\}
\end{align*}
\]

III. Quadratic Formula
\( a = 1 \), \( b = 2 \), \( c = -3 \)
\[
X = \frac{-2 \pm \sqrt{2^2 - 4(1)(-3)}}{2(1)}
\]
\[
X = \frac{-2 \pm \sqrt{4 + 12}}{2}
\]
\[
X = \frac{-2 \pm \sqrt{16}}{2}
\]
\[
X = \frac{-2 \pm 4}{2} \rightarrow X = \frac{-2 + 4}{2}, X = \frac{-2 - 4}{2} \rightarrow X = \frac{2}{2}, X = \frac{-6}{2} \rightarrow \{-3, 1\}\]
Example: Solve the equation using the quadratic formula.

a) \(12x^2 + 5x - 3 = 0\)

\(a = 12, \ b = 5, \ c = -3\)

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
x = \frac{-5 \pm \sqrt{5^2 - 4(12)(-3)}}{2(12)}
\]

\[
x = \frac{-5 \pm \sqrt{25 + 144}}{24}
\]

\[
x = \frac{-5 \pm \sqrt{169}}{24}
\]

\[
x = \frac{-5 \pm 13}{24}
\]

\[
x = \frac{-5 + 13}{24}, \ x = \frac{-5 - 13}{24}
\]

\[
x = \frac{8}{24}, \ x = \frac{-18}{24}
\]

\[
\left\{ -\frac{3}{4}, \frac{1}{3} \right\}
\]
b) \[ 3p^2 = 6p - 1 \]
\[ 3p^2 - 6p + 1 = 0 \]
\[ a = 3, b = -6, c = 1 \]

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(1)}}{2(3)} \]

\[ x = \frac{6 \pm \sqrt{36 - 12}}{6} \]

\[ x = \frac{6 \pm \sqrt{24}}{6} \]

\[ \sqrt{24} = \sqrt{4 \cdot \sqrt{6}} = 2\sqrt{6} \]

\[ x = \frac{6 \pm 2\sqrt{6}}{6} \]

\[ x = 1 + \frac{\sqrt{6}}{3}, x = 1 - \frac{\sqrt{6}}{3} \]

\[ \{ 1 - \frac{\sqrt{6}}{3}, 1 + \frac{\sqrt{6}}{3} \} \]
c) \(9m + \frac{4}{m} = 12\)
\[9m^2 + 4 = 12m\]
\[9m^2 - 12m + 4 = 0\]
\[a = 9, \ b = -12, \ c = 4\]

\[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\]

\[x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(9)(4)}}{2(9)}\]

\[x = \frac{12 \pm \sqrt{144 - 144}}{18}\]

\[x = \frac{12 \pm 0}{18}\]

\[x = \frac{12}{18} = \frac{2}{3}\]

\(\{\frac{2}{3}\}\)
d) \( y^2 - 4y + 13 = 0 \)
\[ a = 1, \ b = -4, \ c = 13 \]
\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
\[
x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}
\]
\[
x = \frac{4 \pm \sqrt{16 - 52}}{6}
\]
\[
x = \frac{4 \pm \sqrt{-36}}{2}
\]
\[
\sqrt{-36} = \sqrt{36} \cdot i = 6i
\]
\[
x = \frac{4 \pm 6i}{2}
\]
\[
x = 2 + 3i, \ x = 2 - 3i
\]
\[
\{2 - 3i, 2 + 3i\}
\]

Discriminant
\( b^2 - 4ac \)

The discriminant can tell us a lot about the number of solutions as well as the type of solutions since the discriminant is the number under the radical in the quadratic formula.

<table>
<thead>
<tr>
<th>Discriminant</th>
<th>Number of Solutions</th>
<th>Type</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive</td>
<td>2</td>
<td>Real</td>
<td>a &amp; b</td>
</tr>
<tr>
<td>Zero</td>
<td>1</td>
<td>Real</td>
<td>c</td>
</tr>
<tr>
<td>Negative</td>
<td>2</td>
<td>Imaginary</td>
<td>d</td>
</tr>
</tbody>
</table>

Example: Find the discriminant of each quadratic equation and indicate whether the equation has two real solutions, one real solution, or two imaginary solutions
\( x^2 - 4x + 2 = 0 \)