The Compound Interest Model

If a principal \( P \) is invested for \( t \) years at an annual rate \( r \) compounded \( n \) times per year, then the amount \( A \), or ending balance, is given by

\[
A = P \left( 1 + \frac{r}{n} \right)^{nt}
\]

The principal \( P \) is also called the present value, and the amount \( A \) is called the future value.

Example 1  Find the future value of $10,000 at 8\% \text{ compounded quarterly for five years.}$

Example 2  Repeat Example 4 for $10,000 at 8\% \text{ but change the frequency of compounding to annually, semi-annually, monthly, daily, and hourly. Compare the future values.}$

<table>
<thead>
<tr>
<th>Frequency of Compounding</th>
<th>( n = ? )</th>
<th>Future value in five years</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annually</td>
<td>$10,000 \left( 1 + \frac{0.08}{1} \right)^{1(5)} $</td>
<td></td>
</tr>
<tr>
<td>Semi-annually</td>
<td>$10,000 \left( 1 + \frac{0.08}{2} \right)^{2(5)} $</td>
<td></td>
</tr>
<tr>
<td>Quarterly</td>
<td>$10,000 \left( 1 + \frac{0.08}{4} \right)^{4(5)} $</td>
<td></td>
</tr>
<tr>
<td>Monthly</td>
<td>$10,000 \left( 1 + \frac{0.08}{12} \right)^{12(5)} $</td>
<td></td>
</tr>
<tr>
<td>Daily</td>
<td>$10,000 \left( 1 + \frac{0.08}{365} \right)^{365(5)} $</td>
<td></td>
</tr>
<tr>
<td>Hourly</td>
<td>$10,000 \left( 1 + \frac{0.08}{8760} \right)^{8760(5)} $</td>
<td></td>
</tr>
</tbody>
</table>
The Continuous Compounding Formula
If a principal \( P \) is invested for \( t \) years at an annual rate \( r \) compounded continuously, then the amount \( A \), or ending balance, is given by \( A = Pe^{rt} \).

**Example 3** Find the future value of $10,000 at 8\% \) compounded continuously for five years.

<table>
<thead>
<tr>
<th>Frequency ((n))</th>
<th>Formula (10000e^{(0.08)(5)} = 10000e^{0.40})</th>
<th>6-year total</th>
</tr>
</thead>
<tbody>
<tr>
<td>continuously ((n = \infty))</td>
<td>$10000e^{(0.08)(5)} = $10000e^{0.40})</td>
<td>$14918.25</td>
</tr>
</tbody>
</table>

**Example 4** If a given amount of money (say $100) is invested in an account that pays compound interest, which would result in a higher rate of return? 6 percent compounded semi-annually or \( \frac{7}{8} \) \% \) compounded daily?

Find the effective rate of interest for each.

6 percent compounded semi-annually
The new balance after 1 year is
\[
\left( 1 + \frac{0.06}{2} \right)^2 = 106.09
\]
The effective rate of interest is 6.09\%.

\( \frac{7}{8} \) \% \) compounded daily
The new balance after 1 year is
\[
100 \left( 1 + \frac{0.05875}{365} \right)^{365} = 106.05
\]
The effective rate of interest is 6.05\%.

When interest is compounded a finite number of times per year, the effective rate of interest can be calculated using the formula \( \left( 1 + \frac{r}{n} \right)^n - 1 \), where \( r \) is the stated rate or nominal rate.

6 percent compounded semi-annually
\[
\left( 1 + \frac{0.06}{2} \right)^2 - 1 = 0.0609 \text{ or } 6.09\%
\]
The effective rate of interest is 6.09\%.

\( \frac{7}{8} \) \% \) compounded daily
\[
\left( 1 + \frac{0.05875}{365} \right)^{365} = 0.0605 \text{ or } 6.05\%
\]
The effective rate of interest is 6.05\%.

**Present Value**

How much money \( P \) should be invested now at a fixed rate of interest \( r \), in order to have a balance of \( A \), \( t \) years from now?

To determine this, we can solve the formula \( A = P \left( 1 + \frac{r}{n} \right)^{nt} \) for the variable \( P \).

**Example 5** How much money should be deposited in an account paying 6\% annual interest compounded monthly in order to have a balance of $200,000 after 10 years?
Time Required to Double or Triple the Value of an Investment

Example 6 If $1000 is invested at an 8% annual interest rate and interest is compounded quarterly, how long will it take for the balance of the account to double? Express your answer in years and quarters of a year. Round to the nearest quarter.

\[ A = P\left(1 + \frac{r}{n}\right)^{nt} \]
\[ A = 2000 \]
\[ P = 1000 \]
\[ r = 0.08 \]
\[ n = 4 \]
\[ t = ? \]

\[ \frac{\ln 2}{4 \ln 1.02} = \frac{(4 \ln 1.02)t}{4 \ln 1.02} \]

Example 7 Suppose that we invest $1000 in an account that compounds interest continuously. What interest rate must the account pay if it doubles the balance in the account every 5 years? Round to the nearest tenth of a percent.

\[ A = Pe^{rt} \]
\[ A = 2000 \]
\[ P = 1000 \]
\[ t = 5 \]

\[ \ln 2 = 5r \]
\[ \ln 2 \approx 0.13863 \text{ or } 13.9\% \]