Steps for Solving Polynomial and Rational Inequalities Algebraically

Step 1: Write the inequality so that a polynomial or rational expression \( f \) is on the left side and zero is on the right side in one of the following forms:

\[ f(x) > 0 \quad f(x) \geq 0 \quad f(x) < 0 \quad f(x) \leq 0 \]

For rational expressions, be sure that the left side is written as a single quotient. Use the form \( \frac{x-3}{x+5} > 0 \) rather than \( \frac{1}{8} > 0 \).

Step 2: Determine the numbers at which the expression \( f \) on the left side equals zero and, if the expression is rational, the numbers at which the expression \( f \) on the left side is undefined.

Step 3: Use the numbers found in Step 2 to separate the real number line into intervals.

Step 4: Select a test number \( c \) in each interval and evaluate \( f \) at the test number.

(a) If the value of \( f(c) \) is positive, then \( f(x) > 0 \) for all numbers \( x \) in the interval.

(b) If the value of \( f(c) \) is negative, then \( f(x) < 0 \) for all numbers \( x \) in the interval.

If the inequality is not strict, include the solutions of \( f(x) = 0 \) in the solution set. Be careful, however. Test numbers at which \( f \) is not defined are not solutions to \( f(x) = 0 \) and should not be included in the solution set.

Example 1 Solve \( x^2 \geq x + 6 \).

Step 1 Rearrange the inequality so that 0 is on the right side.

Steps 2, 3 Solve \( x^2 - x - 6 = 0 \) to find the zeros of \( f(x) = x^2 - x - 6 \).

Draw a real number line and place these numbers on it. Do not place 0 on the number line because it is not a zero of \( x^2 - x - 6 \).

The zeros of \( x^2 - x - 6 \) separate the real number line into three intervals.

The solution set to the inequality \( x^2 - x - 6 \geq 0 \) is made up of one or more of these intervals together with the zeros themselves.

Choose a test number from each interval and determine whether each number makes the expression \( x^2 - x - 6 \) positive or negative.

Step 4 The solution set is the union of the intervals whose test numbers made the expression \( x^2 - x - 6 \) positive. Since the inequality was not strict, the zeros of \( x^2 - x - 6 \) are also included in the solution set. Express the solution set in interval notation.
Example 2 Solve $x^3 - 3x < 18 - 4x^2$.

Example 3 Solve $\frac{7}{x - 1} \geq 2$.

Example 4 Solve $\frac{3}{x - 2} + \frac{4}{x + 1} \geq 0$.

Example 5 Find the domain of the function

$$f(x) = \sqrt{36 - x^2}.$$