Definition of a polynomial function:
Let \( n \) be a nonnegative integer and let \( a_n, a_{n-1}, \ldots, a_2, a_1, a_0 \), be real numbers with \( a_n \neq 0 \).
A polynomial function is a function that can be written in the form
\[
 f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0
\]
The number \( a_n \), the coefficient of the term having the highest power, is called the leading coefficient.
The term \( a_n x^n \), the term with the highest degree, is called the leading term.
The exponent \( n \) on the leading term is called the degree of the polynomial.

Example 1

<table>
<thead>
<tr>
<th>Degree</th>
<th>Type</th>
<th>Example</th>
<th>Sample Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>( f(x) = 3 )</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>( f(x) = 2x + 3 )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>( f(x) = 2x^2 - 5x - 6 )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>( f(x) = x^3 + 2x^2 - 5x - 1 )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>( f(x) = x^4 + 10x^3 - 2x^2 - 120x + 50 )</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>( f(x) = x^5 + 3x^4 - 15x^3 - 19x^2 + 30x + 20 )</td>
<td></td>
</tr>
</tbody>
</table>
Example 2 Which of the following functions are polynomial functions? For those that are polynomial functions, indicate the degree and the leading coefficient.

a. \( f(x) = 3x^2 - \sqrt{x} \)  

b. \( g(x) = \frac{x^2 - 7}{2x + 1} \)

c. \( h(x) = 6 - 3x + 14x^2 \)  

d. \( F(x) = (2x - 3)(x + 4)(5x - 1) \)

Turning Points
Points at the peak of a graph where the function stops increasing and begins to decrease or at the bottom of a valley where the function stops decreasing and begins to increase are called turning points.

Example 3 Find the turning points of the function \( f(x) = 10x^3 - 2x^2 - 120x + 50 \). Round the coordinates of each point using two decimal places.

The number of turning points of the graph of a polynomial function of degree \( n \geq 1 \) is at most \( n - 1 \). Therefore, if the graph of a polynomial has as many as \( n - 1 \) turning points, its degree must be at least \( n \).
Example 4 Graph each 5th degree polynomial function and indicate how many turning points each one has.

a. \( f(x) = 4x^5 + 15x^4 - 100x^3 - 190x^2 + 600x \)  
   b. \( g(x) = 4x^5 + 15x^4 - 100x^3 + 190x^2 + 600x \)

The graph of \( f \) has _____ turning points. The graph of \( g \) has _____ turning points.

Polynomial Functions of the form \( f(x) = ax^n \)

Example 5 Use your calculator to compare the graphs of \( y = x, y = x^2, y = x^3, \) and \( y = x^4 \).

Example 6 What happens when the right-hand side of each graph in Example 1 is multiplied by \(-1\)? Graph the equations \( y = -x, y = -x^2, y = -x^3, \) and \( y = -x^4 \).
**Example 7a** Find a polynomial function with zeros 4, 2, and −3, where 4 is a zero of multiplicity 1, 2 is a zero of multiplicity 3 and −3 is a zero of multiplicity 2. What is the degree of the polynomial that you found?

\[ f(x) = (x - 4)(x - 2)^3(x + 3)^2 \]

If a polynomial is written as the product of polynomial factors, the degree of the polynomial is equal to the sum of the degrees of its factors.

Many different polynomial functions have the properties indicated in the above example. Below are the graphs of two others.

\[ f(x) = 2(x - 4)(x - 2)^3(x + 3)^2 \]

\[ f(x) = -(x - 4)(x - 2)^3(x + 3)^2 \]

Each one can be written in the form \( f(x) = a(x - 4)(x - 2)^3(x + 3)^2 \) where \( a \) is some nonzero constant.
Example 7b  Find the particular polynomial function with zeros 4, 2, and −3, where 4 is a zero of multiplicity 1, 2 is a zero of multiplicity 3, −3 is a zero of multiplicity 2, and whose y-intercept is the point (0, 126).

\[
f(x) = a(x - 4)(x - 2)^3(x + 3)^2
\]

\[
f(0) = a((0) - 4)((0) - 2)^3((0) + 3)^2 = 126
\]

\[a(-4)(-2)^3(3)^2 = 126\]

\[a(-4)(-8)(9) = 126\]

\[\frac{288a}{288} = \frac{126}{288}\]

\[a = \frac{7}{16}\]

The polynomial function is \(f(x) = \frac{7}{16}(x - 4)(x - 2)^3(x + 3)^2\).

Example 8  Find each zero of the polynomial function \(f(x) = 5x(x - 2)^4(x + 1)^3\) and state its multiplicity. What is the degree of this polynomial?

\[f(x) = 5x(x - 2)^4(x + 1)^3\]
End Behavior

For large values of $|x|$, with $x$ either positive or negative, the graph of the polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

resembles the graph of the power function $y = a_n x^n$.

<table>
<thead>
<tr>
<th></th>
<th>$a_n$ is positive</th>
<th>$a_n$ is negative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$ is even</td>
<td><img src="image1" alt="Graph" /></td>
<td><img src="image2" alt="Graph" /></td>
</tr>
<tr>
<td>$n$ is odd</td>
<td><img src="image3" alt="Graph" /></td>
<td><img src="image4" alt="Graph" /></td>
</tr>
</tbody>
</table>
Example 10 Find the leading term, degree, and leading coefficient of each polynomial function. Use an end behavior diagram to indicate the far-left and far-right behavior of each polynomial function.

a. \( f(x) = 3x^4 - 16x^3 + 10x - 5 \)  
b. \( f(x) = 2x^5 - 81x^4 + 320x + 4 \)

c. \( f(x) = (-5x + 9)(3x - 3)^2(2x - 7)^3 \)  
d. \( f(x) = (4 - 3x)(2x + 5)^3(x - 700) \)

Example 11 Which of the following polynomials could be the graph of \( f(x) = -x^5 - 3x^4 + 15x^3 + 19x^2 - 30x + 100 \)?