Definition of Geometric Sequence

A sequence is geometric if the ratios of consecutive terms are the same.

\[
\frac{a_2}{a_1} = r, \quad \frac{a_3}{a_2} = r, \quad \frac{a_4}{a_3} = r, \quad \ldots \quad r \neq 0
\]

The number \( r \) is called the common ratio of the sequence.

A geometric sequence can be defined recursively as follows:

\[
a_1 = a; \quad a_n = ra_{n-1}
\]

Example 1

a. The sequence whose \( n^{th} \) term is \( 3 \left( \frac{1}{2} \right)^n \) is geometric.
   The first few terms are \( \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \ldots \)
   What is the common ratio?

b. The sequence whose \( n^{th} \) term is \( 5(2^n) \) is geometric.
   The first few terms are 10, 20, 40, 80, ...
   What is the common ratio?

c. The sequence whose \( n^{th} \) term is \( 2(-3)^n \).
   The first few terms are -6, 18, -54, 162, ...
   What is the common ratio?

Example 2  Find the first five terms of the geometric sequence whose first term is \( \frac{2}{3} \) and whose common ratio is 3.
The \( n \)th Term of a Geometric Sequence

The \( n \)th term of a geometric sequence has the form

\[ a_n = a_1 r^{n-1} \]

where \( r \) is the common ratio of consecutive terms of the sequence. Thus, every geometric sequence can be written in the following form:

\[ a_1, a_1 r, a_1 r^2, a_1 r^3, \ldots \]

**Example 3** Find the 10th term of the geometric sequence whose first term is 1024 and whose common ratio is \( \frac{1}{2} \).

**Example 4** Find the 10th term of the geometric sequence \( \frac{3}{8}, \frac{3}{4}, \frac{3}{2}, \ldots \).

The common ratio is not given, so we must find it.

\[ r = \frac{a_2}{a_1} = \frac{\frac{3}{4}}{\frac{3}{8}} = \frac{3}{4} \cdot \frac{8}{3} = \frac{2}{1} = 2 \]

Now we can use the first term \( a_1 = \frac{3}{8} \) and the common ratio \( r = 2 \) to write and expression for the \( n \)th term.

\[ a_n = a_1 r^{n-1} \]

\[ a_n = \left( \frac{3}{8} \right) (2)^{n-1} \]

Now we can let \( n = 10 \) to find the 10th term.

\[ a_{10} = \left( \frac{3}{8} \right) (2)^{10-1} = \frac{3}{8} (2)^9 = \frac{3}{8} (512) = 192 \]

The Sum of a Finite Geometric Sequence

The sum of the finite geometric sequence \( a_1, a_1 r, a_1 r^2, \ldots, a_1 r^{n-1} \) with common ratio \( r \neq 1 \) is given by \( S_n = a_1 \left( \frac{1 - r^n}{1 - r} \right) \).
Example 5  Suppose Joshua agrees to work all day for $0.01 on November 1, for twice that much on November 2, twice that much on November 3, and so on, doubling his daily wages each day until November 30. What is the total amount of money that he will make during the month of November?

\[ a_1 = 0.01; \quad r = 2; \quad n = 30 \]

\[ S_{30} = a_1 \left( \frac{1 - r^{30}}{1 - r} \right) = 0.01 \left( \frac{1 - 2^{30}}{1 - 2} \right) \approx \]

The problem can be solved without the formula by using the calculator to evaluate the following sum:

\[ \sum_{n=1}^{30} (0.01)(2^{n-1}) = sum(seq(0.01 * 2^(n-1), n, 1, 30, 1)) \approx \]

Example 6  If Samantha invests $100.00 at the end of each month for 48 months in a savings account that pays interest at a 6% annual rate compounded monthly, what will be the total balance in her account at the end of that time?

\[ a_1 = 100; \quad r = 1.005; \quad n = 48 \]

<table>
<thead>
<tr>
<th>Deposit is made at the end of month #</th>
<th>Number of months that this money is on deposit</th>
<th>Actual contribution to total balance at the end of the 48 months ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>48</td>
<td>0</td>
<td>100.00</td>
</tr>
<tr>
<td>47</td>
<td>1</td>
<td>100(1 + \frac{0.06}{12})^1 = 100(1.005) = 100.50</td>
</tr>
<tr>
<td>46</td>
<td>2</td>
<td>100(1.005)^2 = 101.00</td>
</tr>
<tr>
<td>2</td>
<td>46</td>
<td>100(1.005)^{46} = 125.79</td>
</tr>
<tr>
<td>1</td>
<td>47</td>
<td>100(1.005)^{47} = 126.42</td>
</tr>
</tbody>
</table>

\[ \sum_{n=1}^{48} (100)(1.005)^{n-1} = sum(seq(100 * 1.005^{n-1}, n, 1, 48, 1)) \approx \]

\[ S_{48} = a_1 \left( \frac{1 - r^{48}}{1 - r} \right) = 100 \left( \frac{1 - (1.005)^{48}}{1 - 1.005} \right) \approx \]
Geometric Series

The sum of the terms of an infinite sequence is called an infinite series. If $|r| < 1$, the infinite geometric series $a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$ has the value $S_{\infty} = \sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1 - r}.$

Example 7 Find the value of each infinite geometric series.

a. $\sum_{k=1}^{\infty} 8 \left( \frac{2}{5} \right)^{k-1}$

b. $32 - 24 + 18 - 13.5 + \ldots$

c. $\sum_{k=1}^{\infty} 100(1.005)^{k-1}$