Definition of Arithmetic Sequence

A sequence is arithmetic if the differences between consecutive terms are the same. Thus, the sequence

\[ a_1, a_2, a_3, \ldots, a_n, \ldots \]

is arithmetic if there is a number \( d \) such that

\[ a_2 - a_1 = d, \quad a_3 - a_2 = d, \quad a_4 - a_3 = d, \quad \text{and so on.} \]

The number \( d \) is called the common difference of the arithmetic sequence.

An arithmetic sequence can be also be defined recursively as follows.

\[ a_1 = a \]
\[ a_n = a_{n-1} + d. \]

Example 1

a. The sequence whose \( n^{th} \) term is \( 2n + 5 \) is arithmetic.

\[ 7, 9, 11, 13, \ldots, 2n + 5, \ldots \]

What is the common difference? \( d = \)

Define this sequence recursively. \( a_1 = 7; \ a_n = \)

b. The sequence whose \( n^{th} \) term is \( -3n + 5 \) is arithmetic.

The terms are \( 2, -1, -4, -7, \ldots, -3n + 5, \ldots \)

What is the common difference? \( d = \)

Define this sequence recursively. \( a_1 = 2; \ a_n = \)
The $n^{th}$ Term of an Arithmetic Sequence

If we would like to know the first few terms of an arithmetic sequence, a recursive definition will do. However, if we are interested in knowing the one-thousandth term, for example, it would be nice if we could find it without first having to compute the first 999 terms. So, let’s look for a pattern.

\[
a_2 = a_1 + d \\
a_3 = a_1 + 2d \\
a_4 = a_1 + 3d \\
a_5 = a_1 + 4d \\
a_n = a_1 + (n - 1)d
\]

**Example 2** Find an expression for the $n^{th}$ term of each of the following arithmetic sequences.

a. The first term is 40 and the common difference is 7.

\[
a_1 = 40 \quad \text{and} \quad d = 7 \\
a_n = a_1 + (n - 1)d \\
a_n = (40) + (n - 1)(7) \\
a_n = 40 + 7n - 7 \\
a_n = 7n + 33
\]

b. The first term is 18 and the common difference is -3.

\[
a_1 = 18 \quad \text{and} \quad d = -3 \\
a_n = a_1 + (n - 1)d \\
a_n = (18) + (n - 1)(-3) \\
a_n = 18 - 3n + 3 \\
a_n = -3n + 21
\]

If two terms of an arithmetic sequence are known, say $a_m$ and $a_n$, the common difference can be found by subtracting the equation $a_m = a_1 + (m - 1)d$ from the equation $a_n = a_1 + (n - 1)d$.

\[
a_n = a_1 + (n - 1)d \\
- a_m = a_1 - (m - 1)d \\
a_n - a_m = (n - m)d
\]

\[
a_n - a_m = (n - m)d \\
\Rightarrow (n - m)d = a_n - a_m \\
\Rightarrow d = \frac{a_n - a_m}{n - m}
\]
If you know the common difference and any other term, you can find the first term.

\[ a_n = a_1 + (n - 1)d \]

\[ a_1 = a_n - (n - 1)d \]

However, you may find it more convenient to memorize an alternate formula for

\[ a_n \] in terms of an arbitrary term \( a_m \) rather than having to find the first term \( a_1 \).

From above, we had the equation \( a_n - a_m = (n - m)d \).

If we add \( a_m \) to both sides, we easily obtain the following useful formula:

\[ a_n = a_m + (n - m)d \]

It is fairly easy to remember once you memorize \( a_n = a_1 + (n - 1)d \).

Just replace the number 1 with the letter \( m \) in the two places it occurs.

**Example 3** Find an expression for the \( n^{th} \) term of each of the following arithmetic sequences.

a. The tenth term is 100 and the fortieth term is 280.

\[ a_8 = 40 \text{ and } a_{30} = 128 \]

*First, find the common difference \( d \)* using the formula \( d = \frac{a_n - a_m}{n - m} \).

\[ d = \frac{a_{30} - a_8}{30 - 8} = \frac{128 - 40}{30 - 8} = \frac{88}{22} = 4 \]

*Then, find an expression for the \( a_n \)* using the formula \( a_n = a_m + (n - m)d \).

\[ a_n = a_8 + (n - 8)(4) = 40 + 4n - 32 = 4n + 8 \]

Check it! \( a_8 = 4(8) + 8 = 32 \)

\[ a_{30} = 4(30) + 8 = 128 \]

b. The eighth term is 40 and the thirtieth term is 128.

\[ a_8 = 40 \text{ and } a_{30} = 128 \]

*First, find the common difference \( d \)* using the formula \( d = \frac{a_n - a_m}{n - m} \).

\[ d = \frac{a_{21} - a_9}{21 - 9} = \frac{116 - 20}{21 - 9} = \frac{88}{22} = 4 \]

*Then, find an expression for the \( a_n \)* using the formula \( a_n = a_m + (n - m)d \).

\[ a_n = a_9 + (n - 9)(8) = 20 + 8n - 72 = 8n - 52 \]

\[ a_{800} = 8(800) - 52 = 6348 \]

**Example 4** Find the \( 800^{th} \) term of the sequence whose \( 9^{th} \) term is 20 and whose \( 21^{st} \) term is 116.
The Sum of \( n \) Terms of an Arithmetic Sequence
The sum of the first \( n \) terms of an arithmetic sequence is given by

\[
S_n = n \left( \frac{a_1 + a_n}{2} \right) \quad \text{or} \quad \frac{n}{2}(a_1 + a_n)
\]

Proof

\[
\begin{align*}
S_n &= a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_1 + (n-1)d) \\
S_n &= a_n + (a_n - d) + (a_n - 2d) + \ldots + (a_n - (n-1)d)
\end{align*}
\]

\[
2S_n = (a_1 + a_n) + (a_1 + a_n) + \ldots + (a_1 + a_n)
\]

\[
2S_n = n(a_1 + a_n)
\]

\[
S_n = n \left( \frac{a_1 + a_n}{2} \right)
\]

Example 5 Find \( S_7 \) if \( S_7 = 6 + 9 + 12 + 15 + 18 + 21 + 24 \).

Example 6 Using a graphing utility, find the sum of the first 100 terms of the arithmetic sequence with first term 6 and common difference 4.

Without a graphing utility, you would need to find the first term \( (a_1 = 6) \), the last term \( (a_{100} = ?) \), and the number of terms \( (100) \). To find the last term, it would help to have an expression for \( a_n \).

\[
a_n = a_1 + (n-1)d
\]

\[
a_n = 6 + (n-1)4
\]

\[
a_n = 4n + 2. \quad \text{Now you can find} \ a_{100}.
\]

\[
a_{100} = 4(100) + 2 = 402
\]

Finally, use the formula \( S_n = n \left( \frac{a_1 + a_n}{2} \right) \).

\[
S_{100} = 100 \left( \frac{a_1 + a_{100}}{2} \right)
\]

\[
S_{100} = 100 \left( \frac{6 + 402}{2} \right) = 100(204) = 20400
\]

With a graphing utility, the task is somewhat simpler. All you really need is an expression for \( a_n \).

\[
a_n = 4n + 2
\]

\[
\text{sum(seq}(4n+2,n,1,100,1)) = 20400
\]

\[
4n + 2 \quad \text{is the expression}
\]

\[
n \quad \text{is the variable}
\]

The first 1 is the lower limit. The 100 is the upper limit. The second 1 indicates the increment.

Type this: \( \text{2nd LIST} > > 5: \text{sum(} \)