Definition of a Sequence

A sequence is a function whose domain is set of positive integers. The elements of the range are called the terms of the sequence. Subscript notation rather than function notation is used to denote the terms of a sequence. For example, the first three terms would be denoted \(a_1, a_2,\) and \(a_3\) rather than \(a(1), a(2),\) and \(a(3)\). The TI-83 calculators are an exception to this rule. They use the letters \(u, v,\) and \(w\) to denote sequences, and they use the function notation \(u(n)\) rather than \(u_n\) as we would if we were writing the sequence on paper.

If the domain of a sequence consists of the first \(n\) positive integers only, the sequence is called a finite sequence. Many sequences are defined by giving an expression for the \(n^{th}\) term:

- e.g. \(a_n = 2n + 1\).
- Others, such as the Fibonacci sequence found on page 938 are most easily defined recursively (in terms of previous terms of the sequence).

**Example 1** Find the first four terms of the sequence given by \(a_n = \frac{(n)(n + 1)}{2}\).

**Example 2** Find the first four terms of the sequence given by \((-1)^{n-1}\left(\frac{n + 1}{n}\right)\).
Example 3 The population of the winter moth has been modeled by the recursively-defined sequence, where \(a_n\) represents the population density in thousands per acre during year \(n\).

\[
a_1 = a \\
a_n = 2.85a_{n-1} - 0.19a_{n-1}^2, \text{ for } n \geq 2
\]

The TI83 and TI84 calculators use the names \(u, v, \text{ and } w\) for sequences.

With the calculator in sequence mode, hit \(y=\) and enter the following information:

- \(nMin = 1\)
- \(u(n) = 2.85u(n-1) - 0.19u(n-1)^2\)
- \(u(nMin) = 0.05\)

Type \(2nd \ TBLSET\) and enter the following:

- \(TblStart = 1\)
- \(\Delta Tbl = 1\)
- \(Indpnt : Auto\)
- \(Depend : Auto\)

Complete each table below:

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_n)</td>
<td>0.05</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_n)</td>
<td>10.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_n)</td>
<td>15.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Summation Notation

The sum of the first \(n\) terms of a sequence is represented by

\[
\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n
\]

where \(i\) is called the index of summation, \(n\) is the upper limit of summation, and \(1\) is the lower limit of summation.

Example 4 Write out the terms of each sum.

a. \[
\sum_{i=1}^{4} i^2
\]

b. \[
\sum_{k=3}^{5} (k^2 + 3k - 4)
\]
A finite series is an expression of the form 
\[ S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i. \]

and an infinite series is an expression of the form 
\[ S_\infty = a_1 + a_2 + a_3 + \ldots a_n + \ldots = \sum_{i=1}^{\infty} a_i. \]

Example 5 Write each series using summation notation.

a. 1 + 4 + 7 + 10 + \ldots + 22 
   b. \( \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots + \frac{1}{121} \)

c. \( \frac{1}{2} - \frac{3}{4} + \frac{5}{8} - \frac{7}{16} + \frac{9}{32} - \ldots \) 
   d. 1 + 2 + 3 + 4 + 5 + \ldots 

Properties of Sums and Sequences

If \{a_n\} and \{b_n\} are sequences, and \(c\) is a real number, then:

1. \( \sum_{k=1}^{n} c = nc \) or \( cn \)

2. \( \sum_{k=1}^{n} ca_k = c \sum_{k=1}^{n} a_k \)

3. \( \sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \)

4. \( \sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k \)

5. \( \sum_{k=1}^{n} a_k = \sum_{k=1}^{j} a_k + \sum_{k=j+1}^{n} a_k, \) where \(1 < j < n\)

6. \( \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \)

7. \( \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \)

8. \( \sum_{k=1}^{n} k^3 = \left( \frac{n(n+1)}{2} \right)^2 \)
Example 6 Use the properties above to express the following sum as a polynomial in the variable n.

\[
\sum_{k=1}^{n} (12k^2 - 8k + 5) = \sum_{k=1}^{n} 12k^2 - \sum_{k=1}^{n} 8k + \sum_{k=1}^{n} 5
\]

\[
= 12 \sum_{k=1}^{n} k^2 - 8 \sum_{k=1}^{n} k + 5 \sum_{k=1}^{n} 1
\]

\[
= 12\left(\frac{n(n+1)(2n+1)}{6}\right) - 8\left(\frac{n(n+1)}{2}\right) + 5n
\]

\[
= 2n(n+1)(2n+1) - 4n(n+1) + 5n
\]

\[
= 2n(2n^2 + 3n + 1) - 4n^2 - 4n + 5n
\]

\[
= 4n^3 + 6n^2 + 2n - 4n^2 + n
\]

\[
= 4n^3 + 2n^2 + 3n
\]