A system of equations in two variables is a collection of two or more equations in two variables that are considered at the same time. A solution to a system of equations in two variables is an ordered pair \((x, y)\) that satisfies all equations in the system.

\[
\begin{cases}
  x + y = 7 \\
  y = x^2 + 1
\end{cases}
\]

is an example of such a system. Its solutions are \((-3, 10)\) and \((2, 5)\).

Each equation determines a set of points (a graph) in a two-dimensional rectangular coordinate system. If the graphs of the two equations intersect, the points of intersection are ordered pairs which satisfy both equations. Since these ordered pairs are solutions to both equations, they are the solutions to the system. In this section we will explore two methods of finding solutions to systems of equations: the method of substitution and a graphical method.

The Method of Substitution

Our textbook outlines a 5-step procedure for solving a system of equations by substitution. I have made a few minor modifications.

Example 1 Solve \(\begin{cases} x + y = 7 \\
                      y = x^2 + 1 \end{cases}\) using the substitution method.

1. Solve one of the equations for one variable in terms of the other.

   \[
   \begin{align*}
   x + y &= 7 \\
   y &= -x + 7
   \end{align*}
   \]

2. Substitute the expression found in Step 1 into the other equation to obtain an equation in one variable.

   \[
   \begin{align*}
   y &= x^2 + 1 \\
   (-x + 7) &= x^2 + 1
   \end{align*}
   \]

3. Solve the equation obtained in Step 2.

   \[
   \begin{align*}
   (-x + 7) &= x^2 + 1 \\
   -x + 7 &= x^2 + 1 \\
   0 &= x^2 + x - 6 \\
   0 &= (x + 3)(x - 2) \\
   x &= -3 \text{ or } x = 2
   \end{align*}
   \]

4. Back-substitute the solution(s) in Step 3 into the expression obtained in Step 1 to find the corresponding value(s) of the other variable.

   \[
   \begin{align*}
   y &= -x + 7 & y &= -x + 7 \\
   y &= -(-3) + 7 & y &= -(2) + 7 \\
   y &= 10 & y &= 5
   \end{align*}
   \]

   So, \((-3, 10)\) is a solution. So, \((2, 5)\) is another solution.

5. Check that the solution satisfies each of the original equations.

   \[
   \begin{align*}
   (-3) + (10) &= 7 \\
   (10) &= (-3)^2 + 1 \\
   (2) + (5) &= 7 \\
   (5) &= (2)^2 + 1
   \end{align*}
   \]
Example 2 Solve the system \[ \begin{cases} x^2 + (y - 1)^2 = 25 \\ y - x = 2 \end{cases} \] using the substitution method.

1. Solve one of the equations for one variable in terms of the other.
   \[ y = x + 2 \]

2. Substitute the expression found in Step 1 into the other equation to obtain an equation in one variable.
   \[ x^2 + (y - 1)^2 = 25 \]
   \[ x^2 + ((x + 2) - 1)^2 = 25 \]

3. Solve the equation obtained in Step 2.
   \[ x^2 + ((x + 2) - 1)^2 = 25 \]
   \[ x^2 + (x + 1)^2 = 25 \]
   \[ x^2 + x^2 + 2x + 1 = 25 \]
   \[ 2x^2 + 2x - 24 = 0 \]
   \[ 2(x^2 + x - 12) = 0 \]
   \[ 2(x + 4)(x - 3) = 0 \]
   \[ x = -4 \quad \text{or} \quad x = 3 \]

4. Back-substitute the solution(s) in Step 3 into the expression obtained in Step 1 to find the corresponding value(s) of the other variable.
   \[ x = -2 \quad \text{or} \quad x = 3 \]
   \[ y = x + 2 \]
   \[ y = (-4) + 2 \]
   \[ y = -2 \]
   \[ y = (3) + 2 \]
   \[ y = 5 \]

   So, \((-4, -2)\) is a solution. So, \((3, 5)\) is another solution.

5. Check that the solution satisfies each of the original equations.
   \[ (-4)^2 + ((-2) - 1)^2 = 25 ? \]
   \[ (3)^2 + ((5) - 1)^2 = 25 ? \]
   \[ (-2) - (-4) = 2 ? \]
   \[ (5) - (3) = 2 ? \]
Example 3 Solve the system \[ \begin{align*} x^2 + y^2 &= 13 \\ y &= x^2 - 7 \end{align*} \] using the substitution method.

Example 4 Solve the system \[ \begin{align*} x + y &= x^2 \\ x + 2y &= 6 \end{align*} \] graphically.
Example 5 Solve the system \[
\begin{align*}
  x^2 + (y - 1)^2 &= 25 \\
  y - x &= 2
\end{align*}
\] graphically.

Example 6 Solve the system \[
\begin{align*}
  x^2 + y^2 &= 9 \\
  y &= x^2 - 5
\end{align*}
\] graphically.