MAC 1140 11.4 Matrix Algebra

**Definition:** If $m$ and $n$ are positive integers, an $m \times n$ (read "$m$ by $n$") **matrix** is a rectangular array

\[
\begin{bmatrix}
  a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  a_{i1} & a_{i2} & a_{ij} & a_{in} & \ddots & \vdots \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \ldots & a_{mj} & \ldots & a_{mn}
\end{bmatrix}
\]

in which each entry $a_{ij}$ of the matrix is a real number. An $m \times n$ matrix has $m$ rows (horizontal lines of numbers) and $n$ columns (vertical lines of numbers). The entry in the $i^{th}$ row and $j^{th}$ column is denoted by the double subscript notation $a_{ij}$. A matrix having $m$ rows and $n$ columns is said to be of order $m \times n$. If $m = n$, the matrix is **square** of order $n$.

**Example 1**

Let \( A = \begin{bmatrix} 5 & -3 & 10 & 2 \\ 6 & 8 & -9 & 4 \\ 13 & 21 & 0 & -7 \end{bmatrix} \)

What is the order of matrix $A$?

What entries are indicated below?

\[
\begin{align*}
  a_{11} &= & a_{12} &= & a_{13} &= & a_{14} &= \\
  a_{21} &= & a_{22} &= & a_{23} &= & a_{24} &= \\
  a_{31} &= & a_{33} &= & a_{33} &= & a_{34} &=
\end{align*}
\]
Example 2  The matrix \[
\begin{bmatrix}
4 \\
7 \\
9
\end{bmatrix}
\] is called a column matrix because it has only one column.
What is its order?

Example 3  The matrix \[
\begin{bmatrix}
9 & 3 & 2
\end{bmatrix}
\] is called a row matrix because it has only one row.
What is its order?

Example 4  The matrix \[
\begin{bmatrix}
4 & 7 & 8 \\
3 & 2 & 5 \\
9 & 1 & 6
\end{bmatrix}
\] is called a square matrix because it has the same number of rows as columns. What is its order?

Equality of Matrices
Two matrices are equal if they have the same order and all of their corresponding entries are equal.

Example 5  Which of the following matrices is equal to \[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}
\]? Duh.

a. \[
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{bmatrix}
\]  b. \[
\begin{bmatrix}
1 & 4 \\
2 & 5 \\
3 & 6
\end{bmatrix}
\]  c. \[
\begin{bmatrix}
1 & 3 & 5 \\
2 & 4 & 6
\end{bmatrix}
\]  d. \[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}
\]

Example 6  If \[
\begin{bmatrix}
3 & x & 6 \\
y & 7 & 2
\end{bmatrix}
= \begin{bmatrix}
3 & 1 & 6 \\
y^2 & 7 & 2
\end{bmatrix}
\], find x and y.
Matrix Addition and Subtraction and Scalar Multiplication

You can only add matrices of the same order.
You can add two matrices of the same order by adding their corresponding entries.

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both matrices of order $m \times n$, then their sum $A + B$ is the $m \times n$ matrix given by

$$A + B = [a_{ij} + b_{ij}]$$

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both matrices of order $m \times n$, then their difference $A - B$ is the $m \times n$ matrix given by

$$A - B = [a_{ij} - b_{ij}]$$

Example 7

a. $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 6 & 9 \end{bmatrix} + \begin{bmatrix} -3 & 9 & 6 \\ -2 & 0 & 4 \end{bmatrix} =$

b. $\begin{bmatrix} 1 & 2 \\ 9 & 3 \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 3 & 8 \end{bmatrix} =$

c. $\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 & 6 \end{bmatrix} =$

Scalar multiplication

When an $m \times n$ matrix $A$ is multiplied by a number $k$, each entry of the product $kA$ is formed by multiplying $k$ by the corresponding entry of matrix $A$.

If $A = [a_{ij}]$ is an $m \times n$ matrix and $k$ is a scalar, the scalar multiple of $A$ by $k$ is the $m \times n$ matrix given by

$$kA = [ka_{ij}]$$

Example 8 Let $A = \begin{bmatrix} 2 & 0 \\ 4 & 5 \end{bmatrix}$ and let $B = \begin{bmatrix} 6 & -3 \\ 1 & -2 \end{bmatrix}$. Find $2A - 3B$. 
Properties of Matrix Addition and Scalar Multiplication

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<tr>
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<tbody>
<tr>
<td>1.</td>
<td>$A + B = B + A$</td>
<td>Commutative Property of Matrix Addition</td>
</tr>
<tr>
<td>2.</td>
<td>$A + (B + C) = (A + B) + C$</td>
<td>Associative Property of Matrix Addition</td>
</tr>
<tr>
<td>3.</td>
<td>$(cd)A = c(dA)$</td>
<td>Associative Property of Scalar Multiplication</td>
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<td>4.</td>
<td>$1A = A$</td>
<td>Scalar Identity</td>
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<tr>
<td>5.</td>
<td>$c(A + B) = cA + cB$</td>
<td>Distributive Property</td>
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<tr>
<td>6.</td>
<td>$(c + d)A = cA + dA$</td>
<td>Distributive Property</td>
</tr>
</tbody>
</table>

The order $m \times n$ matrix in which each entry is zero is called the $m \times n$ zero matrix. It serves as the additive identity for the set of all $m \times n$ matrices.

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$ is the additive identity for the set of all $3 \times 2$ matrices.

If $A$ is any $3 \times 2$ matrix, then $A + O = A$ and $O + A = A$.

Matrix Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, the product $AB$ is the $m \times p$ matrix $[c_{ij}]$ where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \ldots + a_{in}b_{nj}.$$ 

That is, the entry $c_{ij}$ in the product matrix $AB$ is formed by multiplying each term of row $i$ of matrix $A$ by the corresponding term of row $j$ of matrix $B$ and adding the results.
Example 9 Let \( A = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \) and let \( B = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} \).

Find the product \( AB \).

Example 10 Let \( A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ 0 & 5 \end{bmatrix} \) and let \( B = \begin{bmatrix} 4 & -2 & 5 \\ 1 & 3 & 6 \end{bmatrix} \).

Find \( AB \) and \( BA \) if possible.

Example 11 A gas station has 4 pumps. At each pump, one can purchase unleaded gasoline for $2.299 per gallon, unleaded plus at $2.399 per gallon, and premium unleaded at $2.499 per gallon. On one particular day, the following amounts of each grade of gasoline were purchased at these pumps as indicated.

<table>
<thead>
<tr>
<th></th>
<th>Regular Unleaded</th>
<th>Unleaded Plus</th>
<th>Premium Unleaded</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pump 1</td>
<td>728.3</td>
<td>179.8</td>
<td>214.5</td>
</tr>
<tr>
<td>Pump 2</td>
<td>616.2</td>
<td>255.9</td>
<td>301.1</td>
</tr>
<tr>
<td>Pump 3</td>
<td>359.6</td>
<td>102.3</td>
<td>255.0</td>
</tr>
<tr>
<td>Pump 4</td>
<td>477.3</td>
<td>199.4</td>
<td>271.2</td>
</tr>
</tbody>
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Let \( A = \begin{bmatrix} 728.3 & 179.8 & 214.5 \\ 616.2 & 255.9 & 301.1 \\ 359.6 & 102.3 & 255.0 \\ 477.3 & 199.4 & 271.2 \end{bmatrix} \) and let \( B = \begin{bmatrix} 2.299 \\ 2.399 \\ 2.499 \end{bmatrix} \).

Use your calculator to verify that the product below is indeed \( AB \), with each entry rounded to the nearest hundredth.
Properties of Matrix Multiplication
Let $A$, $B$, and $C$ be matrices and let $c$ be a scalar.

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<tbody>
<tr>
<td>1. $A(BC) = (AB)C$</td>
<td>Associative Property of Matrix Multiplication</td>
<td></td>
</tr>
<tr>
<td>2. $A(B + C) = AB + AC$</td>
<td>Distributive Property</td>
<td></td>
</tr>
<tr>
<td>3. $(A + B)C = AC + BC$</td>
<td>Distributive Property</td>
<td></td>
</tr>
<tr>
<td>4. $c(AB) = (cA)B = A(cB)$</td>
<td>Associative Property of Scalar Multiplication</td>
<td></td>
</tr>
</tbody>
</table>

Important note: Even if $A$ and $B$ are square matrices of the same order, $AB$ does not necessarily equal $BA$.

\[
\begin{bmatrix}
4 & 7 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
6 & 4 \\
8 & 9
\end{bmatrix}
= 
\begin{bmatrix}
80 & 79 \\
20 & 17
\end{bmatrix}
\]

\[
\begin{bmatrix}
6 & 4 \\
8 & 9
\end{bmatrix}
\begin{bmatrix}
4 & 7 \\
2 & 1
\end{bmatrix}
= 
\begin{bmatrix}
32 & 46 \\
50 & 65
\end{bmatrix}
\]

That is, matrix multiplication is not a commutative operation.

The $n \times n$ matrix that consists of 1’s on its main diagonal and 0’s elsewhere is called the **identity matrix of order** $n$ and is denoted by

\[
I_n = 
\begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{bmatrix}
\]

If $A$ is an $n \times n$ matrix, $I_n$ has the property that $AI_n = A = I_nA$.

For example
\[
\begin{bmatrix}
3 & -6 \\
4 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
= 
\begin{bmatrix}
3 & -6 \\
4 & 0
\end{bmatrix}
\]

and
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
3 & -6 \\
4 & 0
\end{bmatrix}
= 
\begin{bmatrix}
3 & -6 \\
4 & 0
\end{bmatrix}.
\]
The $n \times n$ matrix that consists of 1's on its main diagonal and 0's elsewhere is called the **identity matrix of order** $n$ and is denoted by

$$I_n = \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{bmatrix}$$

If $A$ is an $n \times n$ matrix, $I_n$ has the property that $AI_n = A = I_n A$.

We have already seen that a system of linear equations can be represented by a matrix. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix}$, let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and let $B = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$.

Show that the matrix equation $AX = B$ is equivalent to the system

$$\begin{cases}
x - 2y + z = -4 \\
y + 2z = 4 \\
2x + 3y - 2z = 2
\end{cases}.$$

We have seen that a system of linear equations such as

$$\begin{cases}
x + 2y = -4 \\
2x + 9y + z = 4 \\
4y + z = 2
\end{cases}$$

could be expressed as the matrix equation $AX = B$

using the matrices $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 9 & 1 \\ 0 & 4 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $B = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$.

Solving this system would be very easy if we had a matrix that we could multiply by $A$ to obtain the identity matrix $I$. We would call such a matrix the **inverse** of the matrix $A$, we would denote this matrix by the symbol $A^{-1}$ (read "$A$ inverse"), and it would have the following properties: $A^{-1} A = I$ and $AA^{-1} = I$.

Since it has these properties, we would use it as follows to solve $AX = B$.

$$AX = B$$

$$A^{-1} AX = A^{-1} B$$

$$IX = A^{-1} B$$

$$X = A^{-1} B$$
Example 12
Verify that
\[
\begin{bmatrix}
5 & -2 & 2 \\
-2 & 1 & -1 \\
8 & -4 & 5
\end{bmatrix}
\]
is an inverse for the matrix
\[
A = \begin{bmatrix}
1 & 2 & 0 \\
2 & 9 & 1 \\
0 & 4 & 1
\end{bmatrix}.
\]

Example 13
Use the inverse
\[
A^{-1} = \begin{bmatrix}
5 & -2 & 2 \\
-2 & 1 & -1 \\
8 & -4 & 5
\end{bmatrix}
\]
to solve the matrix equation
\[
AX = B
\]
where
\[
A = \begin{bmatrix}
1 & 2 & 0 \\
2 & 9 & 1 \\
0 & 4 & 1
\end{bmatrix},
X = \begin{bmatrix}
x \\
y \\
z
\end{bmatrix},
\text{and } B = \begin{bmatrix}
-4 \\
4 \\
2
\end{bmatrix}.
\]
Example 14 Find the inverse of the matrix \[
\begin{bmatrix}
 1 & 3 \\
 2 & 5 \\
\end{bmatrix}
\] or show that one does not exist. To do this, put the matrix \[
\begin{bmatrix}
 1 & 3 & 1 & 0 \\
 2 & 5 & 0 & 1 \\
\end{bmatrix}
\] into reduced row-echelon form.

Example 15 Find the inverse of the matrix \[
A = \begin{bmatrix}
 1 & 2 & 0 \\
 2 & 9 & 1 \\
 0 & 4 & 1 \\
\end{bmatrix}
\]. Begin by adjoining the identity matrix \[
I = \begin{bmatrix}
 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1 \\
\end{bmatrix}
\]. to \(A\) to form \([A : I]\).

\[
[A : I] = \begin{bmatrix}
 1 & 2 & 0 & 1 & 0 & 0 \\
 2 & 9 & 1 & 0 & 1 & 0 \\
 0 & 4 & 1 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Now, perform row operations on this 3 x 6 matrix until it is in reduced row-echelon form. If the first three columns are the same as the columns of the 3 x 3 identity matrix, the last three columns are those of \(A^{-1}\), and the resulting matrix will be of the form \([I : A^{-1}]\). Otherwise, the first three entries of at least one row of the 3 x 6 matrix will be zeros, in which case the matrix \(A\) does not have an inverse.

Example 16 Show that the matrix \[
A = \begin{bmatrix}
 1 & 3 & 4 \\
 3 & 5 & 2 \\
 5 & 11 & 10 \\
\end{bmatrix}
\] has no inverse.