MAC 1140 11.3 Systems of Linear Equations: Determinants

Definition of the Determinant of a 2 x 2 Matrix

The determinant of the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given by

$$det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$ 

Example 1 Find the determinant of each 2 × 2 matrix below.

a. $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$  
b. $B = \begin{bmatrix} 3 & -1 \\ 4 & 5 \end{bmatrix}$  
c. $C = \begin{bmatrix} 6 & 9 \\ 4 & 6 \end{bmatrix}$

Finding the Determinant of a 3 x 3 Matrix

The determinant of the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is given by

$$det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$ 

The Minors of a Square Matrix

The 2 × 2 determinants

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

used above to find the determinant of the 3 × 3 matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

are called minors of that 3 × 3 matrix.
If $A$ is a square matrix, the minor $M_{ij}$ of the entry $a_{ij}$ is the determinant of the matrix obtained by deleting row $i$ and column $j$ of matrix $A$.

**Example 2** Find all minors for the matrix $\begin{bmatrix} 4 & 2 & 3 \\ 2 & 7 & 4 \\ 5 & 8 & 6 \end{bmatrix}$.

That is, find $M_{11}$, $M_{12}$, and $M_{13}$.

\[
M_{11} = \begin{vmatrix} 7 & 4 \\ 8 & 6 \end{vmatrix} = 10 \\
M_{12} = \begin{vmatrix} 4 & 5 \\ 6 & 6 \end{vmatrix} = -8 \\
M_{13} = \begin{vmatrix} 2 & 7 \\ 5 & 8 \end{vmatrix} = -19
\]

\[
M_{21} = \begin{vmatrix} 2 & 3 \\ 8 & 6 \end{vmatrix} = -12 \\
M_{22} = \begin{vmatrix} 4 & 3 \\ 5 & 6 \end{vmatrix} = 9 \\
M_{23} = \begin{vmatrix} 4 & 2 \\ 5 & 8 \end{vmatrix} = 22
\]

\[
M_{31} = \begin{vmatrix} 2 & 3 \\ 7 & 4 \end{vmatrix} = -13 \\
M_{32} = \begin{vmatrix} 4 & 3 \\ 2 & 4 \end{vmatrix} = 10 \\
M_{33} = \begin{vmatrix} 4 & 2 \\ 2 & 7 \end{vmatrix} = 24
\]

**Cofactors**

Let $M_{ij}$ be the minor for element $a_{ij}$ in an $n \times n$ matrix.

The cofactor $A_{ij}$ of $a_{ij}$ is $A_{ij} = (-1)^{i+j}M_{ij}$.

**The Determinant of a Square Matrix of order 3 x 3**

If $A$ is a square matrix (of order 2 x 2 or greater), the determinant of $A$ is the sum of the entries in any row (or column) of $A$ multiplied by their respective cofactors.

Expanding along the first row yields $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$.

Expanding along the second row yields $|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$.

Expanding along the third row yields $|A| = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$.

Expanding along the first column yields $|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$.

Expanding along the second column yields $|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$.

Expanding along the third column yields $|A| = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$.

Applying this definition to find a determinant is called **expanding by cofactors**.
Example 3 Find the determinant of \( \mathbf{A} = \begin{bmatrix} 4 & 2 & 3 \\ 2 & 7 & 4 \\ 5 & 8 & 6 \end{bmatrix} \) by expanding by cofactors along the first row.

\[
|A| = \begin{vmatrix} 4 & 2 & 3 \\ 2 & 7 & 4 \\ 5 & 8 & 6 \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.
\]

\[
= a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12} + a_{13}(-1)^{1+3}M_{13}
\]

\[
= (4)(-1)^2 \begin{vmatrix} 7 & 4 \\ 8 & 6 \end{vmatrix} + (2)(-1)^3 \begin{vmatrix} 4 & 2 \\ 5 & 6 \end{vmatrix} + (3)(-1)^4 \begin{vmatrix} 2 & 7 \\ 5 & 8 \end{vmatrix}
\]

\[
= (4) \begin{vmatrix} 7 & 4 \\ 8 & 6 \end{vmatrix} - (2) \begin{vmatrix} 4 & 2 \\ 5 & 6 \end{vmatrix} + (3) \begin{vmatrix} 2 & 7 \\ 5 & 8 \end{vmatrix}
\]

\[
= (4)(10) - (2)(-8) + (3)(-19)
\]

\[
= -1
\]

The determinant can also be found by expanding along a different row. Expanding along the second row goes like this:

\[
|A| = \begin{vmatrix} 4 & 2 & 3 \\ 2 & 7 & 4 \\ 5 & 8 & 6 \end{vmatrix} = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}.
\]

\[
= a_{21}(-1)^{2+1}M_{21} + a_{22}(-1)^{2+2}M_{22} + a_{23}(-1)^{2+3}M_{23}
\]

\[
= (2)(-1)^{2+1} \begin{vmatrix} 3 & 2 \\ 6 & 8 \end{vmatrix} + (7)(-1)^{2+2} \begin{vmatrix} 3 & 4 \\ 6 & 5 \end{vmatrix} + (4)(-1)^{2+3} \begin{vmatrix} 2 & 4 \\ 5 & 8 \end{vmatrix}
\]

\[
= (2)(-1)^3 \begin{vmatrix} 3 & 2 \\ 6 & 8 \end{vmatrix} + (7)(-1)^4 \begin{vmatrix} 3 & 4 \\ 6 & 5 \end{vmatrix} + (4)(-1)^5 \begin{vmatrix} 2 & 4 \\ 5 & 8 \end{vmatrix}
\]

\[
= -2 \begin{vmatrix} 3 & 2 \\ 6 & 8 \end{vmatrix} + (7) \begin{vmatrix} 4 & 3 \\ 5 & 6 \end{vmatrix} - (4) \begin{vmatrix} 4 & 2 \\ 5 & 8 \end{vmatrix}
\]

\[
= -2(-12) + (7)(9) - (4)(22)
\]

\[
= -1
\]
Or, the determinant can be found by expanding along a column.

Expanding along the first column goes like this:

\[
|A| = \begin{vmatrix} 4 & 2 & 3 \\ 2 & 7 & 4 \\ 5 & 8 & 6 \end{vmatrix} = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}
\]

\[
= a_{11}(-1)^{1+1}M_{11} + a_{21}(-1)^{2+1}M_{21} + a_{31}(-1)^{3+1}M_{31}
\]

\[
= (4)(-1)^{1+1} \begin{vmatrix} 7 & 4 \\ 8 & 6 \end{vmatrix} + (2)(-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 8 & 6 \end{vmatrix} + (5)(-1)^{3+1} \begin{vmatrix} 2 & 3 \\ 7 & 4 \end{vmatrix}
\]

\[
= (4)(-1)^2 \begin{vmatrix} 7 & 4 \\ 8 & 6 \end{vmatrix} - (2)(-1)^3 \begin{vmatrix} 2 & 3 \\ 8 & 6 \end{vmatrix} + (5)(-1)^4 \begin{vmatrix} 2 & 3 \\ 7 & 4 \end{vmatrix}
\]

\[
= (4)(10) - (2)(-12) + (5)(-13)
\]

\[
= -1
\]

**Example 4** Find the determinant of \( A = \begin{bmatrix} 3 & 1 & -3 \\ 6 & -5 & 2 \\ 0 & 3 & 0 \end{bmatrix} \) by expanding by cofactors along a convenient row or column.

**Example 5** Find the determinant of \( A = \begin{bmatrix} 3 & 0 & -3 & 10 \\ 6 & 0 & 2 & 4 \\ 0 & 3 & 0 & -2 \\ 1 & 0 & 5 & 0 \end{bmatrix} \) by expanding by cofactors along a convenient row or column.
Cramer’s Rule

Cramer’s Rule is a method of solving a consistent system of linear equations by expressing the value of each variable in terms of the determinant of the matrix of coefficients.

Use Cramer’s Rule to solve a system of two linear equations and two unknowns.

\[ a_1x + b_1y = c_1 \]
\[ a_2x + b_2y = c_2 \]

Find the determinant of the matrix of coefficients: \( D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \).

If \( D = 0 \), then stop. Cramer’s Rule cannot be used to solve this system.

Find the determinant of the matrix formed by replacing the first column of the matrix of coefficients with the column of constants: \( D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \).

Find the determinant of the matrix formed by replacing the second column of the matrix of coefficients with the column of constants: \( D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \).

If \( D \neq 0 \), then \( x \) and \( y \) are given as follows:

\[ x = \frac{D_x}{D} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{D_y}{D} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \]

Example 6 Use Cramer’s Rule to solve the system \[ \begin{cases} 3x + 2y = -5 \\ 4x + 6y = 7 \end{cases} \].

\[ D = \begin{vmatrix} 3 & 2 \\ 4 & 6 \end{vmatrix} = (3)(6) - (2)(4) = 10, \text{ which is not zero, so Cramer’s Rule will work.} \]

\[ x = \frac{D_x}{D} = \frac{\begin{vmatrix} -5 & 2 \\ 7 & 6 \end{vmatrix}}{10} = \frac{3}{2} \quad \text{and} \quad y = \frac{D_y}{D} = \frac{\begin{vmatrix} 3 & -5 \\ 4 & 7 \end{vmatrix}}{10} = \frac{3}{2} \]
Use Cramer’s Rule to solve a system of three linear equations and three unknowns.

\[ a_1x + b_1y + c_1z = d_1 \]
\[ a_2x + b_2y + c_2z = d_2 \]
\[ a_3x + b_3y + c_3z = d_3 \]

Find the determinant of the matrix of coefficients: \( D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \).

If \( D = 0 \), then stop. Cramer’s Rule cannot be used to solve this system.

Find the determinant of the matrix formed by replacing the first column of the matrix of coefficients with the column of constants: \( D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \).

Find the determinant of the matrix formed by replacing the second column of the matrix of coefficients with the column of constants: \( D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \).

Find the determinant of the matrix formed by replacing the third column of the matrix of coefficients with the column of constants: \( D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \).

If \( D \neq 0 \), then \( x, y, \) and \( z \) are given as follows:

\[ x = \frac{D_x}{D} \]
\[ y = \frac{D_y}{D} \]
\[ z = \frac{D_z}{D} \]
Example 7 Use Cramer’s Rule to solve the system
\[
\begin{align*}
3x + 2z &= 4 \\
x + 2y &= 4 \\
y + z &= 0
\end{align*}
\]

\[
\begin{vmatrix}
3 & 0 & 2 \\
1 & 2 & 0 \\
0 & 1 & 1
\end{vmatrix} = 8, \text{ which is not zero, so Cramer’s Rule will work.}
\]

\[
x = \frac{D_x}{D} = \frac{\begin{vmatrix}
4 & 0 & 2 \\
4 & 2 & 0 \\
3 & 0 & 2
\end{vmatrix}}{\begin{vmatrix}
3 & 0 & 2 \\
1 & 2 & 0 \\
0 & 1 & 1
\end{vmatrix}} = \frac{3\begin{vmatrix}1 & 0 \end{vmatrix} + 2\begin{vmatrix}2 & 0 \end{vmatrix} + 4\begin{vmatrix}2 & 1 \end{vmatrix}}{0}
\]

\[
y = \frac{D_y}{D} = \frac{\begin{vmatrix}3 & 4 & 2 \\
1 & 4 & 0 \\
3 & 0 & 2
\end{vmatrix}}{\begin{vmatrix}
3 & 0 & 2 \\
1 & 2 & 0 \\
0 & 1 & 1
\end{vmatrix}} = \frac{3\begin{vmatrix}4 & 0 \end{vmatrix} + 4\begin{vmatrix}2 & 0 \end{vmatrix} + 2\begin{vmatrix}2 & 1 \end{vmatrix}}{0}
\]

\[
z = \frac{D_z}{D} = \frac{\begin{vmatrix}3 & 0 & 4 \\
1 & 2 & 4 \\
3 & 0 & 2
\end{vmatrix}}{\begin{vmatrix}
3 & 0 & 2 \\
1 & 2 & 0 \\
0 & 1 & 1
\end{vmatrix}} = \frac{3\begin{vmatrix}2 & 4 \end{vmatrix} + 0\begin{vmatrix}2 & 0 \end{vmatrix} + 4\begin{vmatrix}2 & 1 \end{vmatrix}}{0}
\]

Example 8 Use Cramer’s Rule to solve the system
\[
\begin{align*}
3x + 2y - z &= 10 \\
2x - y + 2z &= 5 \\
7x + 3z &= 20
\end{align*}
\]

\[
D = \begin{vmatrix}3 & 2 & -1 \\
2 & -1 & 2 \\
7 & 0 & 3
\end{vmatrix} = 0, \text{ so Cramer’s Rule cannot be used to solve this system.}
\]