Solving a System of Linear Equations by Back-Substitution

This section focuses on solving systems involving three linear equations in which there are three variables. The two systems given below are equivalent, meaning that they have the same solution set. However, the one on the right is much easier to solve.

\[
\begin{align*}
  x + 3y + 2z &= 6 \\
  3x + 10y + 3z &= 27 \\
  2x + 8y - z &= 28
\end{align*}
\]

\[
\begin{align*}
  x + 3y + 2z &= 6 \\
  y - 3z &= 9 \\
  z &= -2
\end{align*}
\]

The solution set to such a system consists of elements of the form \((a, b, c)\) called ordered triples.

The solution set may be interpreted as a subset of three-dimensional space. The graph of each equation in the system is a plane in three-dimensional space. The various different types of solution sets that may arise correspond to the various different ways that three planes may intersect in three-dimensional space.
Example 1  Graph the equation \( x + 3y + 2z = 6 \) in three-dimensional space. Don’t freak out. I will not ask you to do this on the test.

Example 2  Solve the following system of equations by back substitution.

\[
\begin{align*}
x + 3y + 2z &= 6 \\
y - 3z &= 9 \\
z &= -2
\end{align*}
\]

Given the comparative ease with which the system in Example 2 can be solved, our main task in this chapter is to convert linear systems to this form. The process of converting a system not in this form to an equivalent system that is in this form is called Gaussian Elimination or simply, the Method of Elimination.

There are three operations on the equations of a linear system which produce an equivalent system.

(1) Interchange any two equations in the system.
(2) Multiply both sides of one of the equations by a nonzero constant.
(3) Replace an equation by the sum (or difference) of itself and a nonzero multiple of any other equation in the system.

Example 3  Solve the following system of equations by using the Method of Elimination.

\[
\begin{align*}
x + 3y + 2z &= 6 \\
3x + 10y + 3z &= 27 \\
2x + 8y - z &= 28
\end{align*}
\]
Example 4 Solve the following system of equations by using the Method of Elimination.

\[
\begin{align*}
  x - 2y + z &= 2 \\
  2x + y - 3z &= 3 \\
  x - 7y + 6z &= 3
\end{align*}
\]

Example 5 Solve the following system of equations by first using the Method of Elimination.

\[
\begin{align*}
  x - 2y + z &= 2 \\
  2x + y - 3z &= 3 \\
  x - 7y + 6z &= 30
\end{align*}
\]