MAC 1140 Review Problems from Section 4.6

1. A polynomial with real coefficients has zeros 7, 2 + i, and -3i.
   a. What other zeros must it have (if any)?
   b. What is the minimum degree of such a polynomial?
2. Give an example of a third degree polynomial with zeros 3 and 4 - 3i subject to each of the following conditions. Multiply it out completely.
3. Find all zeros, real and imaginary, of the polynomial \( f(x) = 7x^3 - 33x^2 + 111x - 65 \).
4. Find all zeros, real and imaginary, of the polynomial
   \[ f(x) = 5x^5 - 42x^4 + 116x^3 - 80x^2 - 84x + 40, \]
   given that one of its zeros is 3 + i.
5. Given that the polynomial \( f(x) = x^6 - 22x^5 + 190x^4 - 806x^3 + 1817x^2 - 2872x + 4292 \)
   has zeros 2i, 5 + 2i, and 6 + i, how many x -intercepts does its graph have?

Solutions

1. a. Since the coefficients are all real, the imaginary zeros 2 + i and -3i must be paired with their respective complex conjugates 2 - i and 3i. So these must also be zeros.
   b. The polynomial must have at least these four imaginary zeros and the one real zero 7, so its degree must be at least five.
2. a. \( f(x) = (x - 3)(x - (4 - 3i))(x - (4 + 3i)) \)
   \[ f(x) = (x - 3)((x - 4) + 3i)((x - 4) - 3i) \]
   \[ f(x) = (x - 3)((x - 4)^2 - (3i)^2) \]
   \[ f(x) = (x - 3)(x^2 - 8x + 16 - (-9)) \]
   \[ f(x) = (x - 3)(x^2 - 8x + 25) \]
   b. \( f(x) = (x - 3)(x - (4 - 3i))(x - r) \)
   \( r \) may be any number, real or imaginary, except 4 + 3i.
   \( f(x) = (x - 3)(x - (4 - 3i))(x - 1) \) would work nicely.
3. Possibilities for the rational zeros are \( \frac{P}{Q} = \pm 1, \pm 5, \pm 13, \pm 65, \pm \frac{1}{7}, \pm \frac{5}{7}, \pm \frac{13}{7}, \pm \frac{65}{7} \).
   Upon inspection of the graph, it appears that \( \frac{5}{7} \) is the only rational zero.

\[
\begin{array}{c|cccc}
\frac{5}{7} & 7 & -33 & 111 & -65 \\
7 & 5 & -20 & 65 \\
\hline
\end{array}
\]

\[ f(x) = \left(x - \frac{5}{7}\right)(7x^2 - 28x + 91) \]
\[ f(x) = \left(x - \frac{5}{7}\right)(7)(x^2 - 4x + 13) \]
\[ f(x) = (7x - 5)(x^2 - 4x + 13) \]

The remaining two zeros can be found using the quadratic formula to solve the equation \( x^2 - 4x + 13 = 0 \).

\[
x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i
\]
4. All of the coefficients are real, so $3 - i$ must also be a zero.

By the Factor Theorem, $x - (3 + i)$ and $x - (3 - i)$ must both be factors, so their product

$$(x - (3 + i))(x - (3 - i)) = x^2 - 6x + 10$$

is also a factor.

Use polynomial division to divide $5x^5 - 42x^4 + 116x^3 - 80x^2 - 84x + 40$ by $x^2 - 6x + 10$.

$$\frac{5x^5 - 42x^4 + 116x^3 - 80x^2 - 84x + 40}{x^2 - 6x + 10} = 5x^3 - 12x^2 - 6x + 4.$$ 

So, $f(x) = (x^2 - 6x + 10)(5x^3 - 12x^2 - 6x + 4)$

Next, check to see if $5x^3 - 12x^2 - 6x + 4$ may have any rational zeros.

The possibilities for $\frac{p}{q}$ are $\pm 1, \pm 2, \pm 4, \pm \frac{1}{5}, \pm \frac{2}{5}, \pm \frac{4}{5}$.

Upon inspection of the graph, $\frac{2}{5}$ appears to be a rational zero.

\[
\begin{array}{c|cccc}
& 5 & -12 & -6 & 4 \\
\hline
5 & & & & \\
2 & 5 & -10 & -10 & 0 \\
\end{array}
\]

\[
f(x) = (x^2 - 6x + 10)(x - \frac{2}{5})(5x^2 - 10x - 10)
\]

\[
f(x) = (x^2 - 6x + 10)(x - \frac{2}{5})(x^2 - 2x - 2)
\]

\[
f(x) = (x^2 - 6x + 10)(5x - 2)(x^2 - 2x - 2)
\]

The remaining zeros can be found by using the quadratic formula to solve the equation $x^2 - 2x - 2 = 0$.

\[
x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}
\]

In summary, the five zeros are $3 + i, 3 - i, \frac{2}{5}, 1 + \sqrt{3}$, and $1 - \sqrt{3}$.

5. The minimum value of this polynomial in the interval $[-10, 10]$ is about 178.3, but who knows whether or not it might have an $x$–intercept to the right of $x = 10$ or to the left of $x = -10$? You do. You can see that all of the coefficients are real, so the zeros $2i, 5 + 2i, \text{ and } 6 + i$ must be accompanied by their respective complex conjugates: $-2i, 5 - 2i, \text{ and } 6 - i$. That makes six different zeros, and that is as many as a sixth degree polynomial can have. Not one of the six zeros is real, so this polynomial function has no $x$–intercepts.