1. Using the Factor Theorem, determine which of the following polynomials are factors of \( f(x) = 3x^4 + 15x^3 - 156x^2 - 492x + 1440 \).
   \[
   \begin{align*}
   x - 1 & \quad x - 2 & \quad x - 3 & \quad x - 4 & \quad x - 5 & \quad x - 6 & \quad x - 7 & \quad x - 8 & \quad x - 9 & \quad x - 10 \\
   x + 1 & \quad x + 2 & \quad x + 3 & \quad x + 4 & \quad x + 5 & \quad x + 6 & \quad x + 7 & \quad x + 8 & \quad x + 9 & \quad x + 10
   \end{align*}
   \]
   Use this information to write the polynomial \( f(x) \) in factored form. Be careful. Make sure your answer has the correct leading term.

2. List the potential rational zeros of the polynomial function \( f(x) = 6x^3 + 56x^2 + 13x - 36 \). Don’t bother trying to find its actual zeros.

3. Let \( f(x) = 14x^4 - 67x^3 - 10x^2 + 155x - 42 \).
   (a) List the potential rational zeros of \( f \).
   (b) Inspect the graph to determine which, if any, of these potential rational zeros are actually zeros of \( f \).
   (c) Use synthetic division and any rational zeros you may find to factor the polynomial into linear and quadratic factors.
   (d) Use the quadratic formula to find the real zeros of any quadratic factors that may result from this process.
   (e) Express \( f \) so that it is completely factored over the set of real numbers.

4. Find all real solutions of the equation \( 6x^4 - 17x^3 - 56x^2 + 19x + 30 = 0 \).

**Solutions**

1. Graph this polynomial function, and use the \( x \)–intercepts of the graph to give you an idea of where the zeros are. Then evaluate the polynomial at each of these numbers to verify that they are in fact zeros. Also, notice that the polynomial has degree 4. Once you find 4 zeros, there is no need to search for any more.

   \[
   \begin{array}{|c|c|c|c|c|c|}
   \hline
   \text{f(1) = 810} & \text{f(2) = 0} & \text{f(3) = -792} & \text{f(4) = -1296} & \text{f(5) = -1170} \\
   x - 1, \text{ no} & x - 2, \text{ yes} & x - 3, \text{ no} & x - 4, \text{ no} & x - 5, \text{ no} \\
   \hline
   \text{f(6) = 0} & \text{f(7) = 2700} & \text{f(8) = 7488} & \text{f(9) = 14994} & \text{f(10) = 25920} \\
   x - 6, \text{ yes} & x - 7, \text{ no} & x - 8, \text{ no} & x - 9, \text{ no} & x - 10, \text{ no} \\
   \hline
   \text{f(-1) = 1764} & \text{f(-2) = 1728} & \text{f(-3) = 1350} & \text{f(-4) = 720} & \text{f(-5) = 0} \\
   x + 1, \text{ no} & x + 2, \text{ no} & x + 3, \text{ no} & x + 4, \text{ no} & x + 5, \text{ yes} \\
   \hline
   \text{f(-6) = -576} & \text{f(-7) = -702} & \text{f(-8) = 0} & \text{f(-9) = 1980} & \text{f(-10) = 5760} \\
   x + 6, \text{ no} & x + 7, \text{ no} & x + 8, \text{ yes} & x - 9, \text{ no} & x - 10, \text{ no} \\
   \hline
   \end{array}
   \]
We have found that \((x - 2), (x - 6), (x + 5), \) and \((x + 8)\) are all factors of \(f(x)\).

The function \(f\), then, can be written in the following way:
\[
f(x) = a(x - 2)(x - 6)(x + 5)(x + 8)
\]

The leading term of this polynomial is \(a(x)(x)(x) = ax^4\).

The leading term of the given polynomial is \(3x^4\).

It follows that \(a = 3\), so \(f(x) = 3(x - 2)(x - 6)(x + 5)(x + 8)\).

2. If \(\frac{p}{q}\) is a rational zero of \(f\), then \(p\) must be a factor of \(-36\) and \(q\) must be a factor of \(6\).

\[p : \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36 \quad q : \pm 1, \pm 2, \pm 3, \pm 6\]

Please note. The empty boxes are for duplicate entries.

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<th>(\frac{p}{q})</th>
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3. Let \(f(x) = 14x^4 - 67x^3 - 10x^2 + 155x - 42\).

(a) List the potential rational zeros of \(f\).

If \(\frac{p}{q}\) is a rational zero of \(f\), then \(p\) must be a factor of \(-42\) and \(q\) must be a factor of \(14\).

\[p : \pm 1, \pm 2, \pm 3, \pm 6, \pm 7, \pm 14, \pm 21, \pm 42 \quad q : \pm 1, \pm 2, \pm 7, \pm 14\]

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(b) Upon inspection of the graph, the rational zeros are found to be \(-\frac{3}{2}\) and \(\frac{2}{7}\).

There appears to be a real zero between \(x = 1\) and \(x = 2\), but \(\frac{3}{2}\) is the only potential rational zero in this interval, and it turns out not to be a zero of \(f\).

Therefore, the zero between 1 and 2 must be irrational.

There also appears to be a real zero between \(x = 4\) and \(x = 5\). However, since there are no potential rational zeros at all between 4 and 5, this zero also is irrational.
(c) First, use synthetic division with \( a = -\frac{3}{2} \) to write \( f(x) \) with a factor of \( \left(x + \frac{3}{2}\right) \).

\[
\begin{array}{c|ccccc}
-\frac{3}{2} & 14 & -67 & -10 & 155 & -42 \\
21 & 132 & -183 & 42 \\
\hline
14 & -88 & 122 & -28 & 0
\end{array}
\]

\[ f(x) = \left(x + \frac{3}{2}\right)(14x^3 - 88x^2 + 122x - 28) \]

Then, use synthetic division a second time to write \( x - \frac{2}{7} \) as a factor of \( f(x) \) as well.

\[
\begin{array}{c|cccc}
\frac{2}{7} & 14 & -88 & 122 & -28 \\
4 & -24 & 28 \\
\hline
14 & -84 & 98 & 0
\end{array}
\]

\[ f(x) = \left(x + \frac{3}{2}\right)\left(x - \frac{2}{7}\right)(14x^2 - 84x + 98) \]

The next few steps are simply to rid the polynomial of fractions and make it a bit more aesthetically pleasing.

\[ f(x) = \left(x + \frac{3}{2}\right)\left(x - \frac{2}{7}\right)(14)(x^2 - 6x + 7) \]

\[ f(x) = \left(x + \frac{3}{2}\right)\left(x - \frac{2}{7}\right)(2)(7)(x^2 - 6x + 7) \]

\[ f(x) = 2\left(x + \frac{3}{2}\right)(7)(x - \frac{2}{7})(x^2 - 6x + 7) \]

\[ f(x) = (2x + 3)(7x - 2)(x^2 - 6x + 7) \]

(d) The quadratic factor \( x^2 - 6x + 7 \) has two irrational zeros which can be found using the quadratic formula. (Completing the square actually works better for this one, but I will not use it, as it has fallen out of favor among most of today’s College Algebra and Precalculus Algebra students.)

\[ x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(7)}}{2(1)} = \frac{6 \pm \sqrt{36 - 28}}{2} = \frac{6 \pm 2\sqrt{2}}{2} = 3 \pm \sqrt{2} \]

(e) The polynomial \( f(x) \) can be factored completely over the real numbers as follows:

\[ f(x) = (2x + 3)(7x - 2)\left(x - \left(3 + \sqrt{2}\right)\right)\left(x - \left(3 - \sqrt{2}\right)\right) \]
4. Find all real solutions of the equation \(6x^4 - 17x^3 - 56x^2 + 19x + 30 = 0\).

By inspecting the graph while considering the potential rational zeros of this polynomial, it can be found that \(-2\) and \(\frac{5}{6}\) are two rational solutions to the equation.

There may be other rational solutions, but these are enough to get us going with the synthetic division process.

First, use synthetic division with \(a = -2\) to write the left-hand side with a factor of \((x + 2)\).

\[
\begin{array}{c|cccc}
-2 & 6 & -17 & -56 & 19 & 30 \\
 & & 12 & 58 & -4 & -30 \\
--- & --- & --- & --- & --- & --- \\
6 & 6 & -29 & 2 & 15 & 0 \\
\end{array}
\]

\((x + 2)(6x^3 - 29x^2 + 2x + 15) = 0\)

Then, use synthetic division a second time to write \(\left(x - \frac{5}{6}\right)\) as a factor of \(f(x)\) as well.

\[
\begin{array}{c|ccc}
\frac{5}{6} & 6 & -29 & 2 & 15 \\
 & & 5 & -20 & -15 \\
--- & --- & --- & --- & --- \\
6 & 6 & -24 & -18 & 0 \\
\end{array}
\]

\((x + 2)\left(x - \frac{5}{6}\right)(6x^2 - 24x - 18) = 0\)

\((x + 2)\left(x - \frac{5}{6}\right)(6)(x^2 - 4x - 3) = 0\)

\((x + 2)(6x - 5)(x^2 - 4x - 3) = 0\)

To find the two remaining solutions, set \(x^2 - 4x - 3 = 0\) and use the quadratic formula to solve.

\[
x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)} = \frac{4 \pm \sqrt{16 + 12}}{2} = \frac{4 \pm \sqrt{28}}{2} = \frac{4 \pm 2\sqrt{7}}{2} = 2 \pm \sqrt{7}.
\]

The solution set to the original equation is \(\{-2, \frac{5}{6}, 2 + \sqrt{7}, 2 - \sqrt{7}\}\).