For each rational function,

a. Find the domain.
b. Find the $x$-intercept(s).
c. Find the $y$-intercept.
d. Find the vertical asymptotes if there are any.
e. Find any points where the graph has a "hole."
f. Find the nonvertical asymptote if there is one.
g. Find the point where the graph crosses its nonvertical asymptote if there is one.

1. $f(x) = \frac{5x - 15}{x^2 + 2x - 15}$
2. $f(x) = \frac{5x^2 + 28x - 12}{2x^2 + 13x + 15}$
3. $f(x) = \frac{x^3 + 5x^2 - 17x - 21}{x^2 - 3x + 6}$
4. $f(x) = \frac{x^3 - 4x^2 - 11x + 30}{x^2 + x - 6}$

Solutions

1. a. $f(x) = \frac{5x - 15}{x^2 + 2x - 15}$
   
   For the domain, set the denominator equal to zero and solve.
   $x^2 + 2x - 15 = 0$
   $(x + 5)(x - 3) = 0$
   $x + 5 = 0$ or $x - 3 = 0$
   $x = -5$ or $x = 3$
   
   The domain is $\{x | x \neq -5 \text{ and } x \neq 3\}$.
   
   First, make sure the expression is in lowest terms.
   $$\frac{5x - 15}{x^2 + 2x - 15} = \frac{5(x - 3)}{(x + 5)(x - 3)} = \frac{5}{x + 5}$$
   
   So, $f(x) = \frac{5}{x + 5}$, $x \neq 3$
   
   b. To find the $x$-intercepts set $f(x) = 0$ and solve for $x$.
   $$\frac{5x - 15}{x^2 + 2x - 15} = 0$$
   
   A fraction is equal to zero only when its numerator is equal to zero, and its denominator is not zero.
   
   Solve $5x - 15 = 0$
   $5x = 15$
   $x = 3$
   
   Since 3 is not in the domain of $f$, it does not produce an $x$-intercept.
   
   The graph has no $x$-intercepts.
   
   c. To find the $y$-intercept, find $f(0)$.
   $$f(0) = \frac{5(0) - 15}{(0)^2 + 2(0) - 15} = \frac{-15}{-15} = 1.$$ The y-intercept is (0, 1).
d. First, write \( \frac{5x - 15}{x^2 + 2x - 15} \) in lowest terms.

\[
\frac{5x - 15}{x^2 + 2x - 15} = \frac{5(x - 3)}{(x + 5)(x - 3)} = \frac{5}{x + 5}
\]

Solve \( x + 5 = 0 \).

\( x = -5 \) is the only vertical asymptote.

e. Since the function is undefined at \( x = 3 \) even though there is no vertical asymptote at \( x = 3 \), there is a hole in the graph at \( x = 3 \).

The \( y \)-coordinate of the hole can be found by evaluating \( \frac{5}{x + 5} \) at \( x = 3 \).

\[
\frac{5}{3 + 5} = \frac{5}{8}
\]

The hole in the graph is at the point \( \left( 3, \frac{5}{8} \right) \).

f. The degree of the numerator is less than that of the denominator (Case 1), so the \( x \)-axis (the line \( y = 0 \)) is the horizontal asymptote for this function.

The function has no oblique asymptote.

g. Since the \( x \)-axis is the horizontal asymptote for this function, a point where the graph crosses this asymptote would correspond to an \( x \)-intercept in this case, so the graph of this function does not cross its horizontal asymptote.

2. a. To find the domain, do not simplify. Set the original denominator equal to zero and solve for \( x \).

\[
2x^2 + 13x + 15 = 0
\]

\[
(2x + 3)(x + 5) = 0
\]

\[
2x + 3 = 0 \quad \text{or} \quad x + 5 = 0
\]

\( x = -\frac{3}{2} \quad \text{or} \quad x = -5 \)

The domain is \( \left\{ x \mid x \neq -\frac{3}{2} \text{ and } x \neq -5 \right\} \).

b. To find the \( x \)-intercepts, set \( f(x) = 0 \) and solve for \( x \).

\[
\frac{(5x - 2)(x + 6)}{(2x + 3)(x + 5)} = 0
\]

Since a fraction is equal to zero only when its numerator equals zero, the \( x \)-intercepts are the solutions to the equation \( (5x - 2)(x + 6) = 0 \). Since both of these solutions are in the domain of the function, the \( x \)-intercepts are \( \left( \frac{2}{5}, 0 \right) \) and \( (-6, 0) \).

c. To find the \( y \)-intercept of any function \( f \), simplify evaluate \( f(0) \).

\[
f(0) = \frac{5(0)^2 + 28(0) - 12}{2(0)^2 + 13(0) + 15} = \frac{-12}{15} = -\frac{4}{5}
\]

The \( y \)-intercept is \( \left( 0, -\frac{4}{5} \right) \).

d. When searching for vertical asymptotes, first, make sure the rational expression is in lowest terms.

\[
\frac{5x^2 + 28x - 12}{2x^2 + 13x + 15} = \frac{(5x - 2)(x + 6)}{(2x + 3)(x + 5)}
\]

This is in lowest terms.

Since \( \frac{(5x - 2)(x + 6)}{(2x + 3)(x + 5)} \) is in lowest terms, the vertical asymptotes correspond to the real zeros of the denominator.

\( (2x + 3)(x + 5) = 0 \)

The vertical asymptotes are \( x = -\frac{3}{2} \) or \( x = -5 \).
The function \( f(x) = \frac{5x^2 + 28x - 12}{2x^2 + 13x + 15} \) has a vertical asymptote at each zero of its denominator, so it does not have any holes.

Since the degree of the numerator is equal to the degree of the denominator (Case 2), the graph has a horizontal asymptote, the line \( y = \frac{5}{2} \).

To see if the graph actually crosses this horizontal asymptote,

\[
\text{solve the equation } \frac{5x^2 + 28x - 12}{2x^2 + 13x + 15} = \frac{5}{2}.
\]

\[
2(5x^2 + 28x - 12) = 5(2x^2 + 13x + 15)
\]

\[
10x^2 + 56x - 24 = 10x^2 + 65x + 75
\]

\[
56x - 24 = 65x + 75
\]

\[
-9x = 99
\]

\[
x = -11
\]

The graph of \( f \) crosses the line \( y = \frac{5}{2} \) at the point \((-11, \frac{5}{2})\).

3. a. To find the domain, set the denominator equal to zero and solve for \( x \).

\[
x^2 - 3x + 6 = 0
\]

The exact solutions to this equation can be found using the quadratic formula. They are both imaginary. This can be seen simply by using the discriminant.

\[
b^2 - 4ac = (-3)^2 - 4(1)(6) = 9 - 24 = -15
\]

Since the discriminant is negative, the equation has no real solutions. Since no real number makes the denominator zero, all real numbers are in the domain of the function.

b. To find the \( x \)-intercepts, set \( f(x) = 0 \) and solve for \( x \).

\[
\frac{x^3 + 5x^2 - 17x - 21}{x^2 - 3x + 6} = 0
\]

Since a fraction is equal to zero only when its numerator equals zero, the solutions to the equation \( x^3 + 5x^2 - 17x - 21 = 0 \) give us the \( x \)-intercepts.

To solve this, you probably should graph \( y = x^3 + 5x^2 - 17x - 21 \) and check to see whether or not the graph intersects the \( x \)-axis at any of its potential rational zeros: \( \pm 1, \pm 3, \pm 7, \) or \( \pm 21 \)

It appears to intersect the \( x \)-axis at \(-7, -1, \) and \( 3 \).

You should verify that these correspond to \( x \)-intercepts of \( f \) by graphing

\[
\frac{(x^3 + 5x^2 - 17x - 21)}{(x^2 - 3x + 6)} \text{ and using } \boxed{\text{2nd} \ \text{CALC} \ 1 \text{value}}.
\]

The \( x \)-intercepts are the points \((-7, 0), (-1, 0) \) and \( (3, 0) \).

c. Evaluate \( f(0) \) to find the \( y \)-intercept. \( f(0) = \frac{(0)^3 + 5(0)^2 - 17(0) - 21}{(0)^2 - 3(0) + 6} = \frac{-21}{6} = -\frac{7}{2} \)

The \( y \)-intercept is \( \left( 0, -\frac{7}{2} \right) \).

d. Since \( \frac{(x + 7)(x + 1)(x - 3)}{x^2 - 3x + 6} \) is in lowest terms, the vertical asymptotes correspond to the real zeros of the denominator. The denominator has no real zeros, so the graph has no vertical asymptotes.
Since the denominator $x^2 - 3x + 6$ has no real zeros, the domain of $f$ is the set of all real numbers, so its graph does not have any holes.

Since the degree of the numerator is exactly one greater than the degree of the denominator (Case 3), the graph has an oblique asymptote, which can be found by dividing the denominator into the numerator using polynomial division. When this is done, the quotient is $x + 8$, so the oblique asymptote is the line $y = x + 8$.

g. To see if the graph actually crosses this horizontal asymptote, solve the equation
\[
\frac{x^3 + 5x^2 - 17x - 21}{x^2 - 3x + 6} = x + 8
\]
\[
x^3 + 5x^2 - 17x - 21 = (x + 8)(x^2 - 3x + 6)
\]
\[
x^3 + 5x^2 - 17x - 21 = x^3 - 3x^2 + 6x + 8x^2 - 24x + 48
\]
\[
x^3 + 5x^2 - 17x - 21 = x^3 + 5x^2 - 18x + 48
\]
\[
-17x - 21 = -18x + 48
\]
\[
x = 69
\]
The graph of $f$ crosses the line $y = x + 8$ at the point $(69, 77)$.

4. a. To find the domain, solve $x^2 + x - 6 = 0$.
\[
(x + 3)(x - 2) = 0
\]
\[
x = -3 \quad \text{or} \quad x = 2
\]
The domain is \(\{x \mid x \neq -3 \text{ and } x \neq 2\}\).

b. The $x$-intercepts are the zeros of the numerator that are not also zeros of the denominator.
\[
x^3 - 4x^2 - 11x + 30 = 0
\]
\[
(x + 3)(x - 2)(x - 5) = 0
\]
\[
x = -3 \quad \text{or} \quad x = 2 \quad \text{or} \quad x = 5
\]
Since $x = -3$ and $x = 2$ are also zeros of the denominator, they do not produce $x$-intercepts. The other zero gives the $x$-intercept $(5, 0)$.

c. The $y$-intercept is $f(0) = \frac{(0)^3 - 4(0)^2 - 11(0) + 30}{(0)^2 + (0) - 6} = \frac{30}{6} = -5$.

The graph crosses the $y$-axis at the point $(0, -5)$.

d. First, make sure the rational expression is in lowest terms.
\[
f(x) = \frac{x^3 - 4x^2 - 11x + 30}{x^2 + x - 6} = \frac{(x + 3)(x - 2)(x - 5)}{(x + 3)(x - 2)}
\]
can be reduced, and we can write
\[
f(x) = x - 5, \quad \text{so long as we make the restrictions } x \neq -3 \text{ and } x \neq 2.
\]
e. The graph of $f$ is simply the line $y = x - 5$ with two holes in it at the points $(-3, -8)$ and $(2, -3)$.

f. The line $y = x - 5$ is technically the oblique asymptote, but we don’t really need it for the graph.

g. The graph of $f$ intersects it at every point, except for the two holes of course.