MAC 1140 Review Problems from Section 4.1

1. Form a 4th degree polynomial with zeros 3, −2, 4, and 0 whose graph passes through the point (2,40).
2. Form a polynomial with zeros 3, −4, and 2 where 3 is a zero of multiplicity 2, −4 is a zero of multiplicity 3, and 2 is a zero of multiplicity 4. What is the minimum degree of such a polynomial?
3. Find a polynomial whose graph is this graph and whose y-intercept is (0,24).

4. Find the leading term of each polynomial and use it to determine the end behavior of the polynomial.
   a. \( f(x) = 8x^3 - 130x^2 + 19 \)
   b. \( f(x) = x(3x - 2)^2(2x + 5)^3 \)
   c. \( f(x) = -x(2x^2 + x - 1)^2 \)
   d. \( f(x) = 175x^4 - 3x^6 \)
   e. \( f(x) = 7 + 3x^2 - 9x^3 + 10x^5 \)
   f. \( f(x) = (2x + 3)(5x^2 - x + 1)(3x + 11)^3 \)

Solutions

1. The zeros 3, −2, 4, and 0 give us, respectively, the factors \((x - 3), (x + 2), (x - 4), \) and \(x.\) Any 4th degree polynomial with these four factors can be written in the form \( f(x) = a(x - 3)(x + 2)(x - 4)(x) \) for some constant \(a.\)
   Since the graph passes through the point (2,40), we know that \(f(2) = 40.\)
   We can use this information to find the specific value of \(a.\)
   
   \[
   f(2) = a(2 - 3)(2 + 2)(2 - 4)(2) = 40 \\
   a(-1)(4)(-2) = 40 \\
   16a = 40 \Rightarrow a = 2.5
   \]
   So, the polynomial function we want is \(f(x) = 2.5(x - 3)(x + 2)(x - 4)(x).\)

2. The zeros 3, −4, and 2 with their respective multiplicities give us the factors \((x - 3)^2, (x + 4)^3, \) and \((x - 2)^4.\)
   The degrees of these factors are 2, 3, and 4, respectively, so the degree of their product \((x - 3)^2(x + 4)^3(x - 2)^4\) equals the sum of their degrees: 9.
   A polynomial meeting the above criteria may have additional factors which could increase its degree, but its degree must be at least 9.
3. From the looks of the graph it has x-intercepts at -2, 1, and 4.
   The graph crosses the x-axis at x=-2 at a sharp angle, so it appears
   that the multiplicity of this zero is 1.
   The graph crosses the x-axis at x=1, so its multiplicity is odd.
   Since it appears to have a horizontal tangent line at x=1, the
   multiplicity is at least 3. It could be as high as 5 or 7, but we'll call it 3.
   The graph just touches the x-axis at x=4, so the multiplicity of this
   zero is even. It may be as high as 4 or 6 but we'll call it 2.
   With these zeros and their multiplicities, we have the following polynomial.
   \[ f(x) = a(x + 2)(x - 1)^3(x - 4)^2 \]

   Many polynomial functions will have these zeros with these multiplicities,
   but only one has a y-intercept of (0, 24). We can find this particular
   polynomial by finding the value of \( a \).
   \[
   f(0) = a(0 + 2)(0 - 1)^3(0 - 4)^2 = 24 \\
   a(2)(-1)^3(-4)^2 = 24 \\
   a(2)(-1)(16) = 24 \\
   -32a = 24 \\
   \frac{-32a}{-32} = \frac{24}{-32} \\
   a = -\frac{3}{4}
   \]

   The function we are looking for is \( f(x) = -\frac{3}{4}(x + 2)(x - 1)^3(x - 4)^2 \).

4. a. The leading term is \( 8x^3 \).
   The graph eventually goes down to the left and up to the right.

b. The leading term is \( (x)(3x)^2(2x)^3 = 72x^6 \).
   The graph eventually goes up both ways.

c. The leading term is \( (-x)(2x^2)^2 = -4x^5 \).
   The graph eventually goes up to the left and down to the right.

d. The leading term is \( -3x^6 \). The graph eventually goes down both ways.

e. The leading term is \( 10x^5 \). The degree is 5.
   The graph eventually goes down to the left and up to the right.

f. It may be helpful for you to rewrite the polynomial as follows:
   \[ f(x) = (2x + 3)(5x^2 - x + 1)(3x + 11)(3x + 11)(3x + 11) \]
   The degree of the product of these polynomial factors equals the sum of their
degrees.
   The degree of \( f(x) \), then is \( 1 + 2 + 1 + 1 = 6 \).
   The leading term of the product of these polynomial factors is equal to the product
   of their leading terms. The leading term of \( f(x) \) is \( (2x)(5x^2)(3x)(3x)(3x) = 270x^6 \).
   Just for fun, I let Scientific Notebook expand the polynomial completely. Here it is.
   \[ f(x) = 270x^6 + 3321x^5 + 14724x^4 + 27251x^3 + 17105x^2 + 1936x + 3993. \]
   The graph eventually goes up both ways.