1. Consider the piecewise function \( f(x) = \begin{cases} 8x^3 & x < 0 \\ -\sin x & 0 \leq x \end{cases} \). Find \( f\left( f\left( \frac{\pi}{3} \right) \right) \).

\[
f\left( f\left( \frac{\pi}{3} \right) \right) = f\left( -\sin \frac{\pi}{3} \right) = f\left( -\frac{\sqrt{3}}{2} \right) = 8\left( -\frac{\sqrt{3}}{2} \right)^3 = -3\sqrt{3}
\]

Correct Answer: \(-3\sqrt{3}\)

2. A cubic foot is what fraction of a cubic yard?

A foot is equal to \( \frac{1}{3} \) of a yard. A cubic foot is equal to \( \left( \frac{1}{3} \right)^3 = \frac{1}{27} \) of a cubic yard.

Correct Answer: \( \frac{1}{27} \)

3. If \( 3^x - 3^{x-3} = 162 \), then \( x = ? \)

\[
3^x - \frac{3^x}{3} = 162 \quad , \quad \frac{2}{3} \cdot 3^x = 162 \quad , \quad 3^x = 243 = 3^5 \quad , \quad x = 5
\]

Correct Answer: \( 5 \)

4. What is the probability of rolling a 7 at least once in two rolls of a pair of fair, six-sided dice?

The probability of not rolling a 7 is \( \frac{5}{6} \). Therefore, the probability of not rolling a 7 in two rolls is \( \frac{5}{6} \cdot \frac{5}{6} = \frac{25}{36} \). Thus the probability of rolling a 7 at least once in two rolls is \( 1 - \frac{25}{36} = \frac{11}{36} \).

Correct Answer: \( \frac{11}{36} \)
5. Mulletonium doubles in volume every minute. At 09:00 a small amount is placed in a test container, and at 10:00 the container has just reached the full mark. At what time was the container one quarter full?

Since the container is full at 10:00, it is 1/2 full at 09:59 and 1/4 full at 09:58.

Correct Answer: 09:58

6. Given \( f(x) = \frac{x^3}{x^2 + 1} \), what is \( x \) if \( f^{-1}(x) = 2 \)?

\[
x = f(f^{-1}(x)) = f(2) = \frac{2^3}{2^2 + 1} = \frac{8}{5}
\]

Correct Answer: \( \frac{8}{5} \)

7. If \( 2 \tan\left(\cos^{-1}\left(\frac{3}{5}\right)\right) = x \), then \( x = ? \)

Let \( \theta = \cos^{-1}\left(\frac{3}{5}\right) \), \( \cos \theta = \frac{3}{5} \), \( \tan \theta = \frac{4}{3} \), and \( 2 \tan\left(\cos^{-1}\left(\frac{3}{5}\right)\right) = 2 \tan \theta = 2 \cdot \frac{4}{3} = \frac{8}{3} \)

Correct Answer: \( \frac{8}{3} \)

8. A father budgeted $24 to distribute equally among his children for spending money at a pool party. When two young cousins joined the party and shared in equal distribution of this money, each child in the party received $1 less than originally planned. How many children in total were at the party?

Let \( x \) be the number of children in total. Then we have \( \frac{24}{x} = \frac{24}{x-2} - 1 \),
\[ 24(x - 2) = 24x - x(x - 2), \ x^2 - 2x - 48 = 0, \ (x + 6)(x - 8) = 0, \text{ and } x = 8 \]

Correct Answer: 8
9. Determine the numerical value of the following and write the result in simplest, reduced form:

\[
\frac{\tan(90^\circ - x) + \tan(x - 30^\circ)}{1 - \tan(90^\circ - x) \tan(x - 30^\circ)}
\]

\[
\frac{\tan(90^\circ - x) + \tan(x - 30^\circ)}{1 - \tan(90^\circ - x) \tan(x - 30^\circ)} = \tan(90^\circ - x + x - 30^\circ) = \tan 60^\circ = \sqrt{3}
\]

Correct Answer: \(\sqrt{3}\)

10. Who am I?

1736-1813

Hint: "Home, Home, on the Lag _ _ _ _"

Correct Answer: Joseph Lagrange

11. Angle \(x\) is nine times angle \(y\); while the complement of angle \(y\) is nine times the complement of angle \(x\). Find angle \(y\) to the nearest degree.

Solve the system

\[
\begin{align*}
9y &= x \quad (1) \\
90^\circ - y &= 9(90^\circ - x) \quad (2)
\end{align*}
\]

by eliminating \(x\).

Multiplying equation (1) by 9, then add the result to equation (2). We have \(90^\circ + 80y = 810^\circ\). Thus, \(y = 9^\circ\).

Correct Answer: \(9^\circ\)
12. \( \cos\left(\frac{\pi}{2} - x\right) \csc(x) = \) ?

\[
\cos\left(\frac{\pi}{2} - x\right) \csc(x) = \sin x \cdot \frac{1}{\sin x} = 1
\]

Correct Answer: 1

13. Arrange the following three numbers from smallest to largest.

\[
a = \log_2 3, \quad b = \log_{\frac{1}{2}} \frac{1}{5}, \quad c = \log_3 2
\]

\[
a = \log_2 3 = \frac{\ln 3}{\ln 2}, \quad b = \log_{\frac{1}{2}} \frac{1}{5} = \frac{\ln \frac{1}{5}}{\ln \frac{1}{2}} = -\frac{\ln 5}{\ln 2} = -\frac{\ln 5}{\ln 2}, \quad \text{and} \quad c = \log_3 2 = \frac{\ln 2}{\ln 3}.
\]

We have \( c < a < b \).

Correct Answer: \( c < a < b \) or \( \log_3 2 < \log_2 3 < \log_{\frac{1}{2}} \frac{1}{5} \)

14. What is the area of a square whose diagonal is one unit longer than the length of its side?

Let \( x \) be the length of the side of the square. Its diagonal is \( x\sqrt{2} \). We have \( x\sqrt{2} = x + 1 \).

\[
2x^2 = x^2 + 2x + 1, \quad x^2 - 2x = 1, \quad (x-1)^2 = 2, \quad x = 1 + \sqrt{2}, \quad \text{and} \quad x^2 = 3 + 2\sqrt{2}.
\]

Correct Answer: \( 3 + 2\sqrt{2} \) or \( \frac{1}{3 - 2\sqrt{2}} \)

15. If \( x = \cos^2 10^\circ + \cos^2 20^\circ + \cdots + \cos^2 70^\circ + \cos^2 80^\circ \), express the rational number \( x \) in the simplest, reduced form.

\[
\cos^2 10^\circ + \cos^2 20^\circ + \cdots + \cos^2 70^\circ + \cos^2 80^\circ
\]

\[
= \cos^2 10^\circ + \cos^2 20^\circ + \cdots + \cos^2 (90^\circ - 20^\circ) + \cos^2 (90^\circ - 10^\circ)
\]

\[
= \cos^2 10^\circ + \cos^2 20^\circ + \cdots + \sin^2 20^\circ + \sin^2 10^\circ = 1 + 1 + 1 + 1 = 4
\]

Correct Answer: 4
16. Determine the value of the following in simplest form: $a^{\log_a 5}$

$$a^{\log_a 5} = \left( a^{\frac{1}{2}} \right)^{\log_a 5} = (a^2)^{\frac{1}{2} \log_a 5} = (a^2)^{\log_a 5^{1/2}} = 5^{1/2} = \sqrt{5}$$

Correct Answer: $\sqrt{5}$ or $5^{1/2}$

17. Find the fraction that is equal to the repeating decimal $0.201420142014\cdots$.

$$0.201420142014\cdots = \frac{2014}{10000} + \frac{2014}{10000^2} + \frac{2014}{10000^3} + \cdots$$

$$= \frac{2014}{10000} \cdot \frac{1}{1 - \frac{1}{10000}} = \frac{2014}{9999}$$

Correct Answer: $\frac{2014}{9999}$

18. $A$ and $B$ are independent events such that $P(A \mid B) = \frac{1}{2}$ and $P(B \mid A) = \frac{1}{4}$. What is $P(A \cap B)$?

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{2} \quad \text{and} \quad P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{1}{4}.$$  

Then $$\frac{P(A \cap B)}{P(B)} \cdot \frac{P(A \cap B)}{P(A)} = \frac{(P(A \cap B))^2}{P(B)P(A)} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$  

Since $A$ and $B$ are independent events, we have $P(B)P(A) = P(A \cap B)$. Therefore, $P(A \cap B) = \frac{1}{8}$

Correct Answer: $\frac{1}{8}$
19. What is the value of $\sqrt{10 + 4\sqrt{6}} - \sqrt{10 - 4\sqrt{6}}$?

\[
\sqrt{10 + 4\sqrt{6}} - \sqrt{10 - 4\sqrt{6}} \\
= \sqrt{(\sqrt{6})^2 + 2 \cdot 2\sqrt{6} + 2^2} - \sqrt{(\sqrt{6})^2 - 2 \cdot 2\sqrt{6} + 2^2} \\
= \sqrt{(\sqrt{6} + 2)^2} - \sqrt{(\sqrt{6} - 2)^2} = 4
\]

Correct Answer: 4

20. Who am I?

1862 – 1943

Hint: I proposed ten mathematical problems in 1900.

Correct Answer: David Hilbert