Section 4.7: Optimization Problems

General steps:

- Identify all given quantities and all quantities to be determined. If possible, make a sketch.
- Write a primary equation for the quantity that is to be maximized or minimized.
- Write the primary equation in terms of one variable by using a secondary equation. This equation should relate the independent variables in the primary equation.
- Use either the First or Second Derivative tests to find any relative extrema.
- Check the validity of your solutions.

Problems:

1. A rectangular page is to contain 30 square inches of print. The margins on each side are 1 inch. Find the dimensions of the page such that the least amount of paper is used.

\[ P: A = (x+2)(y+2) \]
\[ S: 30 = xy \]
- Solve for \( y \) in \( S \)
  \[ y = \frac{30}{x} \]
- Sub in to \( P \)
  \[ A = (x+2)\left(\frac{30}{x}+2\right) \]
  \[ A = 30 + 2x + \frac{60}{x} + y \]

\[ A = 2x + \frac{60}{x} + 3y \]

Use 2nd Derivative Test

\[ A' = 2 - 60x^{-2} \]
\[ 0 = 2 - 60x^{-2} \]
\[ 60x^{-2} = 2 \]
\[ \frac{60}{x^2} = \frac{2}{1} \]
\[ 30 = x^2 \]
\[ y = \frac{30}{\sqrt{30}} = 5.48 \text{ in} \]

2. A closed rectangular container with a square base is to have a volume of 2250 in\(^3\). The material for the top and bottom of the container will cost $2 per in\(^2\), and the material for the sides will cost $3 per in\(^2\). Find the dimensions of the container of least cost.

\[ P: C = (2)(2)(x^2) + 3(4)(xy) \]
\[ C = 4x^2 + 12xy \]

\[ S: 2250 = x^2y \]
- Solve \( S \) for \( y \)
  \[ \frac{2250}{x^2} = y \]

Plug in \( y \) into \( P \)

Use 2nd Derivative Test

\[ C = 4x^2 + 12x\left(\frac{2250}{x^2}\right) \]
\[ C = 4x^2 + 27000x^{-1} \]
\[ C'(15) = 24 > 0 \]
\[ x = 15 \text{ in} \]
\[ y = \frac{2250}{15^2} = 10 \text{ in} \]

Dimensions:

[15 in x 15 in x 10 in]
3. Suppose that the number of bacteria in a culture at time $t$ is given by 
$$ N(t) = 5000(25 + te^{-t/20}) $$
Find the largest and smallest number of bacteria in the culture during the time interval $0 \leq t \leq 100$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$N(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>125,000 &amp; Min</td>
</tr>
<tr>
<td>20</td>
<td>161,788 &amp; Max</td>
</tr>
<tr>
<td>100</td>
<td>128,364</td>
</tr>
</tbody>
</table>

$N'(t) = 5000(t - \frac{1}{20}e^{-t/20} + e^{-t/20})$

$N'(t) = 5000e^{-t/20}(-\frac{1}{20}t + 1)$

$0 = 5000e^{-t/20}(-\frac{1}{20}t + 1)$

$0 = -\frac{1}{20}t + 1 \Rightarrow t = 20$

4. Find the area of the largest rectangle that can be inscribed under the curve $y = e^{-x^2}$ in the first and second quadrants.

**Diagram:**

**Solution:**

$$ A = 2xe^{-x^2} $$

$$ A' = 2x(-2xe^{-x^2}) + 2e^{-x^2} $$

$$ A' = -4xe^{-x^2} + 2e^{-x^2} $$

$$ A = 2e^{-x^2}(-2x^2 + 1) $$

$$ 0 = -2x^2 + 1 $$

$$ 2x^2 = 1 $$

$$ x^2 = \frac{1}{2} $$

$$ x = \frac{1}{\sqrt{2}} = .71 $$

$$ A = \frac{1}{\sqrt{2}} e^{-1/2} $$

$$ A \approx .86 \text{ units}^2 $$